# ASTB23 – Lecture 19
# Potential - density pairs

Newton's gravity

Spherical systems
- Newtons theorems
- Gauss theorem as an integrated Poisson equation

Simple density distribution and their potentials

Dynamical time

[Below are large portions of Binney and Tremaine textbook's Ch.2.]

## 1 General results

Our goal is to calculate the force $\mathbf{F}(\mathbf{x})$ on a unit mass at position $\mathbf{x}$ that is generated by the gravitational attraction of a distribution of mass $\rho(\mathbf{x})$. According to Isaac Newton's inverse-square law of gravitation, the force $\mathbf{F}(\mathbf{x})$ may be obtained by summing the small contributions

$$\delta\mathbf{F}(\mathbf{x}) = G\frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|^3}\delta m(\mathbf{x}') = G\frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|^3}\rho(\mathbf{x}')\delta^3\mathbf{x}' \qquad (2\text{-}1)$$





to the overall force from each small element of volume $\delta^3\mathbf{x}'$ located at $\mathbf{x}'$. Thus

$$\mathbf{F}(\mathbf{x}) = G\int\frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|^3}\rho(\mathbf{x}')d^3\mathbf{x}'. \qquad (2\text{-}2)$$

If we define the **gravitational potential** $\Phi(\mathbf{x})$ by

$$\Phi(\mathbf{x}) = -G\int\frac{\rho(\mathbf{x}')}{|\mathbf{x}' - \mathbf{x}|}d^3\mathbf{x}', \qquad (2\text{-}3)$$

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and notice that

$$\nabla_{\mathbf{x}}\left(\frac{1}{|\mathbf{x}'-\mathbf{x}|}\right) = \frac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}'-\mathbf{x}|^3}, \tag{2-4}$$

we find that we may write $\mathbf{F}$ as

$$\begin{aligned}\mathbf{F}(\mathbf{x}) &= \nabla_{\mathbf{x}}\int \frac{G\rho(\mathbf{x}')}{|\mathbf{x}'-\mathbf{x}|}d^3\mathbf{x}' \\ &= -\nabla\Phi,\end{aligned} \tag{2-5}$$

where for brevity we have dropped the subscript $\mathbf{x}$ on the gradient operator $\nabla$. Since the force is determined by the gradient of a potential, the gravitational force is conservative (cf. Appendix 1.D.1).

The potential is useful because, being a scalar field, it is easier to visualize than the vector force field. Also, in many situations the best way to obtain $\mathbf{F}$ is first to calculate the potential and then to take its gradient.

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If we take the divergence of equation (2-2), we find

$$\nabla \cdot \mathbf{F}(\mathbf{x}) = G \int \nabla_{\mathbf{x}} \cdot \left( \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|^3} \right) \rho(\mathbf{x}') d^3\mathbf{x}'. \tag{2-6}$$

this page not obligatory, F.Y.I.

Now

$$\nabla_{\mathbf{x}} \cdot \left( \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|^3} \right) = -\frac{3}{|\mathbf{x}' - \mathbf{x}|^3} + \frac{3(\mathbf{x}' - \mathbf{x}) \cdot (\mathbf{x}' - \mathbf{x})}{|\mathbf{x}' - \mathbf{x}|^5}. \tag{2-7}$$

When $\mathbf{x}' - \mathbf{x} \neq 0$ we may cancel the factor $|\mathbf{x}' - \mathbf{x}|^2$ from top and bottom of the last term in this equation to conclude that

$$\nabla_{\mathbf{x}} \cdot \left( \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|^3} \right) = 0 \qquad (\mathbf{x}' \neq \mathbf{x}). \tag{2-8}$$

Therefore, any contribution to the integral of equation (2-6) must come from the point $\mathbf{x}' = \mathbf{x}$, and we may restrict the volume of integration to a small sphere of radius $h$ centered on this point. Since, for

If we substitute from equation (2-5) for $\nabla \cdot \mathbf{F}$, we obtain **Poisson's equation** relating the potential $\Phi$ to the density $\rho$;

$$\nabla^2 \Phi = 4\pi G \rho. \tag{2-10}$$

Equation (2-10) provides a route to $\Phi$, and then to $\mathbf{F}$ that is often more convenient than equation (2-2) or equation (2-3). In the special case $\rho = 0$ we have **Laplace's equation**,

$$\nabla^2 \Phi = 0. \tag{2-11}$$

We may use Poisson's equation to derive a useful generalization of equation (2-8). A unit point mass at $\mathbf{x}'$ has density $\rho(\mathbf{x}) = \delta(\mathbf{x} - \mathbf{x}')$, where $\delta$ is the Dirac delta function [eq. (1C-1)], and potential $-G|\mathbf{x} - \mathbf{x}'|^{-1}$. Hence equation (2-10) yields

$$\nabla^2_{\mathbf{x}} \left( \frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) = -4\pi\delta(\mathbf{x} - \mathbf{x}') \quad \text{or} \quad \nabla_{\mathbf{x}} \cdot \left( \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x} - \mathbf{x}'|^3} \right) = -4\pi\delta(\mathbf{x} - \mathbf{x}'). \tag{2-12}$$

If we integrate both sides of equation (2-10) over an arbitrary volume containing total mass $M$, and then apply the divergence theorem, we obtain

$$4\pi G \int \rho d^3\mathbf{x} = 4\pi G M = \int \nabla^2 \Phi d^3\mathbf{x} = \int \nabla \Phi \cdot d^2\mathbf{S}. \tag{2-13}$$

This result is **Gauss's theorem**, which may be stated in words as *the integral of the normal component of* $\nabla\Phi$ *over any closed surface equals* $4\pi G$ *times the mass contained within that surface.*

We have seen that the gravitational force is conservative, that is, that the work done against gravitational forces in moving two stars from infinity to a given configuration is independent of the path along which they are moved, and is defined to be the potential energy of the configuration. Similarly, the work done against gravitational forces in assembling an arbitrary continuous distribution of mass $\rho(\mathbf{x})$ is independent of the details of how the mass was assembled, and is defined to be equal to the potential energy of the mass distribution or simply the **potential energy**. An expression for the potential energy can be obtained by the following argument.

Suppose that some of the mass is already in place so that the density and potential are $\rho(\mathbf{x})$ and $\Phi(\mathbf{x})$. If we now bring in a small mass $\delta m$ from infinity to position $\mathbf{x}$, the work done is $\delta m \Phi(\mathbf{x})$. Thus, if we add a small increment of density $\delta\rho(\mathbf{x})$, the change in potential energy is

$$\delta W = \int \delta\rho(\mathbf{x})\Phi(\mathbf{x})d^3\mathbf{x}. \qquad (2\text{-}14)$$

According to Poisson's equation the resulting change in potential $\delta\Phi(\mathbf{x})$ satisfies $\nabla^2(\delta\Phi) = 4\pi G(\delta\rho)$, and hence

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$$\delta W = \frac{1}{4\pi G}\int \Phi\nabla^2(\delta\Phi)\, d^3\mathbf{x}. \qquad (2\text{-}15)$$

This derivation will not be, but you must understand the final result

Using the divergence theorem in the form (1B-43), we may write this as

$$\delta W = \frac{1}{4\pi G}\int \Phi\boldsymbol{\nabla}(\delta\Phi)\cdot d^2\mathbf{S} - \frac{1}{4\pi G}\int \boldsymbol{\nabla}\Phi\cdot\boldsymbol{\nabla}(\delta\Phi)d^3\mathbf{x}, \qquad (2\text{-}16)$$

where the surface integral vanishes because $\Phi \propto r^{-1}$, $|\boldsymbol{\nabla}\delta\Phi| \propto r^{-2}$ as $r \to \infty$, so the integrand $\propto r^{-3}$ while the total surface area $\propto r^2$. But $\boldsymbol{\nabla}\Phi\cdot\boldsymbol{\nabla}(\delta\Phi) = \frac{1}{2}\delta|(\boldsymbol{\nabla}\Phi)|^2$. Hence

$$\delta W = -\frac{1}{8\pi G}\delta\left(\int |\boldsymbol{\nabla}\Phi|^2 d^3\mathbf{x}\right). \qquad (2\text{-}17)$$

If we now sum up all of the contributions $\delta W$, we have a simple expression for the potential energy,

$$W = -\frac{1}{8\pi G}\int |\boldsymbol{\nabla}\Phi|^2 d^3\mathbf{x}. \qquad (2\text{-}18)$$

An easy proof of Newton's 1st theorem: re-draw the picture to highlight symmetry, conclude that the angles theta 1 and 2 are equal, so masses of pieces of the shell cut out by the beam are in square relation to the distances $r_1$ and $r_2$. Add two forces, obtain zero vector.

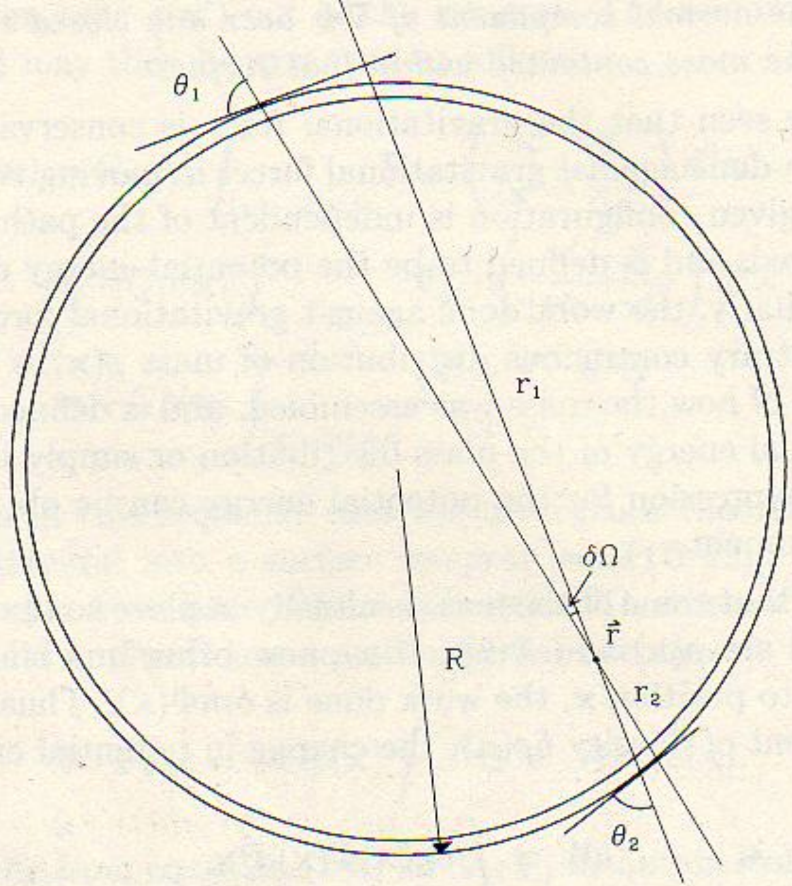


**Figure 2-1.** Proof of Newton's first theorem.

To obtain an alternative expression for $W$, we may again apply the divergence theorem and replace $\nabla^2\Phi$ by $4\pi G\rho$ to obtain

$$W = \frac{1}{2}\int \rho(\mathbf{x})\Phi(\mathbf{x})d^3\mathbf{x}. \quad (2\text{-}19)$$

**Figure 2-1.** Proof of Newton's first theorem.

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$$W = \frac{1}{2}\int \rho(\mathbf{x})\Phi(\mathbf{x})d^3\mathbf{x}. \tag{2-19}$$

## 2.1 Spherical Systems

### 1 Newton's Theorems

Newton proved two results that enable us to calculate the gravitational potential of any spherically symmetric distribution of matter very easily. We may state these as:

**Newton's First Theorem** *A body that is inside a spherical shell of matter experiences no net gravitational force from that shell.*

**Newton's Second Theorem** *The gravitational force on a body that lies outside a closed spherical shell of matter is the same as it would be if all the shell's matter were concentrated into a point at its center.*

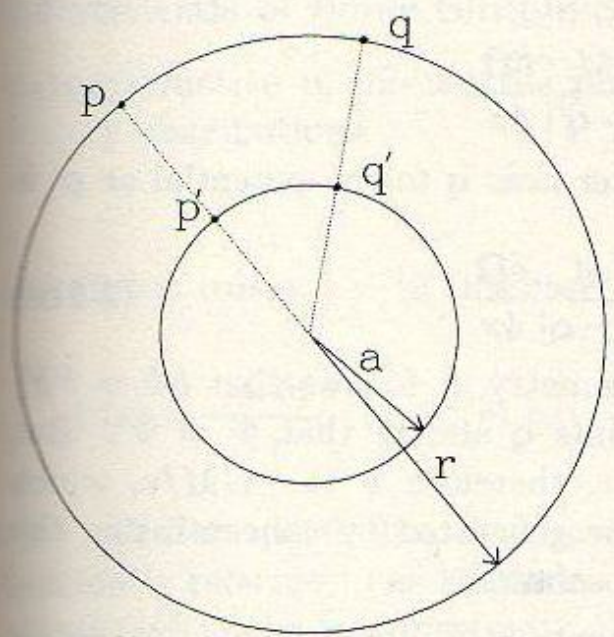


**Figure 2-2.** Proof of Newton's second theorem.

Figure 2-1 illustrates the proof of Newton's first theorem. Consider the cone associated with an elementary solid angle $d\boldsymbol{\Omega}$ centered on the point $\mathbf{r}$. This cone intersects the spherical shell of matter at two points, at distances $r_1$ and $r_2$ from $\mathbf{r}$. Elementary geometrical considerations assure us that the angles $\theta_1$ and $\theta_2$ are equal, and therefore that the masses $\delta m_1$ and $\delta m_2$ contained within $\delta\boldsymbol{\Omega}$ where it intersects the shell are in the ratio $\delta m_1/\delta m_2 = (r_1/r_2)^2$. Hence $\delta m_2/r_2^2 = \delta m_1/r_1^2$ and a particle placed at $\mathbf{r}$ is attracted equally in opposite directions. Summing over all elementary cones centered on $\mathbf{r}$, one concludes that the body at $\mathbf{r}$ experiences no net force from the shell.◁

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An important corollary of Newton's first theorem is that the gravitational potential inside an empty spherical shell is constant because $\nabla\Phi = -\mathbf{F} = 0$. Thus we may evaluate the potential $\Phi(\mathbf{r})$ inside the shell by calculating the integral expression (2-3) for $\mathbf{r}$ located at any interior point. The most convenient place for $\mathbf{r}$ is the center of the shell, for then all points on the shell are at the same distance $R$, and one immediately has

This potential
is per-unit-mass
of the test particle

$$\Phi = -\frac{GM}{R}. \tag{2-20}$$

The proof of Newton's second theorem eluded Newton for more than ten years. Yet with hindsight it is easy. The trick is to compare the potential at a point $\mathbf{p}$ located a distance $r$ from the center of a spherical shell of mass $M$ and radius $a$ ($r > a$), with the potential at a point $\mathbf{p}'$ located a distance $a$ from the center of a shell of mass $M$ and radius $r$. Figure 2-2 illustrates the proof. Consider the contribution $\delta\Phi$ to the potential at $\mathbf{p}$ from the portion of the given sphere with solid angle $\delta\boldsymbol{\Omega}$ located at $\mathbf{q}'$. Evidently

$$\delta\Phi = -\frac{GM}{|\mathbf{p}-\mathbf{q}'|}\frac{\delta\boldsymbol{\Omega}}{4\pi}. \qquad (2\text{-}21a)$$

But the contribution $\delta\Phi'$ of the matter near $\mathbf{q}$ to the potential at $\mathbf{p}'$ is clearly

$$\delta\Phi' = -\frac{GM}{|\mathbf{p}'-\mathbf{q}|}\frac{\delta\boldsymbol{\Omega}}{4\pi}. \qquad (2\text{-}21b)$$

Finally, as $|\mathbf{p}-\mathbf{q}'| = |\mathbf{p}'-\mathbf{q}|$ by symmetry, it follows that $\delta\Phi = \delta\Phi'$, and then by summation over all points $\mathbf{q}$ and $\mathbf{q}'$ that $\Phi = \Phi'$. But we already know that $\Phi' = -GM/r$, therefore $\Phi = -GM/r$, which is exactly the potential that would be generated by concentrating the entire mass of the given sphere at its center.◁

The Newtonian gravitational potentials of different spherical shells add linearly, so we may calculate the gravitational potential at $\mathbf{r}$ generated by an arbitrary spherically symmetric density distribution $\rho(\mathbf{r}')$ in two parts by adding the contributions to the potential produced by shells (i) with $r' < r$, and (ii) with $r' > r$. In this way we obtain

$$\Phi(r) = -4\pi G\left[\frac{1}{r}\int_0^r \rho(r')r'^2 dr' + \int_r^\infty \rho(r')r'dr'\right]. \qquad (2\text{-}22)$$

General solution. Works in all the spherical systems!

Inner & outer shells

From Newton's first and second theorems or from equation (2-22) it follows that the gravitational attraction of the density distribution $\rho(r')$ on a unit test mass at radius $r$ is entirely determined by the mass interior to $r$:

$$\mathbf{F}(r) = -\frac{d\Phi}{dr}\hat{\mathbf{e}}_r = -\frac{GM(r)}{r^2}\hat{\mathbf{e}}_r, \tag{2-23a}$$

where

$$M(r) = 4\pi \int_0^r \rho(r')r'^2 dr'. \tag{2-23b}$$

An important property of a spherical matter distribution is its **circular speed** $v_c(r)$, defined to be the speed of a test particle in a circular orbit at radius $r$. Once we have $\Phi(r)$ or $\mathbf{F}(r)$, we may readily evaluate $v_c$ from

If you can, use the simpler eq. 2-23a for computations

$$v_c^2 = r\frac{d\Phi}{dr} = r|\mathbf{F}| = \frac{GM(r)}{r}. \tag{2-24}$$

The circular speed measures the mass interior to $r$. A second important quantity is the **escape speed** $v_e$ defined by

Potential in this formula must be normalized to zero at infinity!

$$v_e(r) = \sqrt{2|\Phi(r)|}. \tag{2-25}$$

A star at $r$ can escape from the gravitational force field represented by $\Phi$ only if it has a speed at least as great as $v_e(r)$, for only then does its (positive) kinetic energy $\frac{1}{2}v^2$ exceed the absolute value of its (negative) potential energy $\Phi$.

**2 Potentials of Some Simple Systems**

It is instructive to discuss the potentials generated by several simple density distributions:

**(a) Point mass** In this case

$$\Phi(r) = -\frac{GM}{r} \quad ; \quad v_c(r) = \sqrt{\frac{GM}{r}} \quad ; \quad v_e(r) = \sqrt{\frac{2GM}{r}}. \tag{2-26}$$

Any circular speed that declines with increasing radius like $r^{-1/2}$ is frequently referred to as **Keplerian** because Kepler first understood that $v_c \propto r^{-1/2}$ in the solar system.

**(b) Homogeneous sphere** If the density is some constant $\rho$, we have $M(r) = \frac{4}{3}\pi r^3 \rho$ and

(rising rotation curve)

$$v_c = \sqrt{\frac{4\pi G\rho}{3}}\, r. \tag{2-27}$$

Thus in this case the circular velocity rises linearly with radius, and the orbital period of a mass on a circular orbit is

$$T = \frac{2\pi r}{v_c} = \sqrt{\frac{3\pi}{G\rho}}, \tag{2-28}$$

independent of the radius of its orbit.

If a test mass is released from rest at radius $r$ in the gravitational field of a homogeneous body, its equation of motion is

$$\frac{d^2 r}{dt^2} = -\frac{GM(r)}{r^2} = -\frac{4\pi G\rho}{3} r, \tag{2-29}$$

which is the equation of motion of a harmonic oscillator of angular frequency $2\pi/T$. Therefore no matter what is the initial value of $r$, the test mass will reach $r = 0$ in a quarter of a period, or in a time

$$t_{\rm dyn} = \frac{T}{4} = \sqrt{\frac{3\pi}{16 G\rho}}. \tag{2-30}$$

Although this result is only correct for a homogeneous sphere, we shall define the **dynamical time** of a system with mean density $\rho$ by equation

(2-30).[1] The dynamical time is approximately equal to the time required for an orbiting star to travel halfway across a system of this mean density.

From equation (2-22) it follows that if the density vanishes for $r > a$, the gravitational potential is

$$\Phi(r) = \begin{cases} -2\pi G\rho(a^2 - \frac{1}{3}r^2), & r < a \\ -\dfrac{4\pi G\rho a^3}{3r}, & r > a, \end{cases} \tag{2-31}$$

which can be used to compute the escape speed.

Thie definition of dynamical time is NOT universally adopted. Rather, I would like you to remember that most dynamicists consider dynamical time to be the characteristic length scale (radius *r*, if the system is round) divided by characteristic speed (usually circular speed $V_c$): $t_{dyn} = r / V_c$

That means that one orbital period, which is $P = 2\pi r/V_c$, equals $2\pi$~6.28 dynamical times (also called dynamical time scales, or timescales)

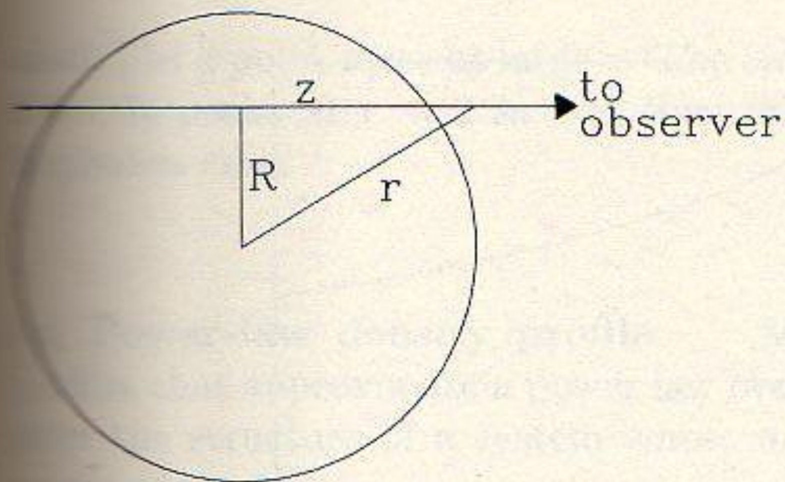



**Figure 2-3.** Projection of a spherical body along the line of sight.

and at large radii the density tends to

$$\rho(r) \simeq \frac{bM}{2\pi r^4} \qquad (r \gg b). \tag{2-36}$$

Know the methods, don't memorize the details of this potential-density pair:

**(d) Modified Hubble profile** The surface brightnesses of many elliptical galaxies may be approximated over a large range of radii by the Hubble-Reynolds law $I_H(R)$ [eq. (1-14)]. It is possible to solve for the spherical luminosity density $j(r)$ that generates a given circularly symelliptical galaxies may be approximated over a large range of radii by the Hubble-Reynolds law $I_H(R)$ [eq. (1-14)]. It is possible to solve for the spherical luminosity density $j(r)$ that generates a given circularly symmetric brightness distribution $I(R)$ (see Problem 2-10). However, the resulting formulae for the luminosity distribution of a galaxy that obeys the Hubble-Reynolds law are cumbersome (Hubble 1930). Fortunately, the simple luminosity density

Spatial density of light

$$j_h(r) = j_0 \left[1 + \left(\frac{r}{a}\right)^2\right]^{-\frac{3}{2}}, \qquad (2\text{-}37)$$

where $a$ is the **core radius**, gives rise to a surface brightness distribution that is similar to $I_H$ (Rood et al. 1972). In fact, in the notation of Figure 2-3 we have that

Surface density of light
on the sky

$$I_h(R) = 2\int_0^\infty j_h(r)\,dz = 2j_0 \int_0^\infty \left[1 + \left(\frac{R}{a}\right)^2 + \left(\frac{z}{a}\right)^2\right]^{-\frac{3}{2}} dz. \qquad (2\text{-}38)$$

Using the substitution $y \equiv z/\sqrt{a^2+R^2}$, we obtain the **modified Hubble profile**

$$I_h(R) = \frac{2j_0 a}{1+(R/a)^2}\int_0^\infty \frac{dy}{(1+y^2)^{\frac{3}{2}}} = \frac{2j_0 a}{1+(R/a)^2}. \qquad (2\text{-}39)$$

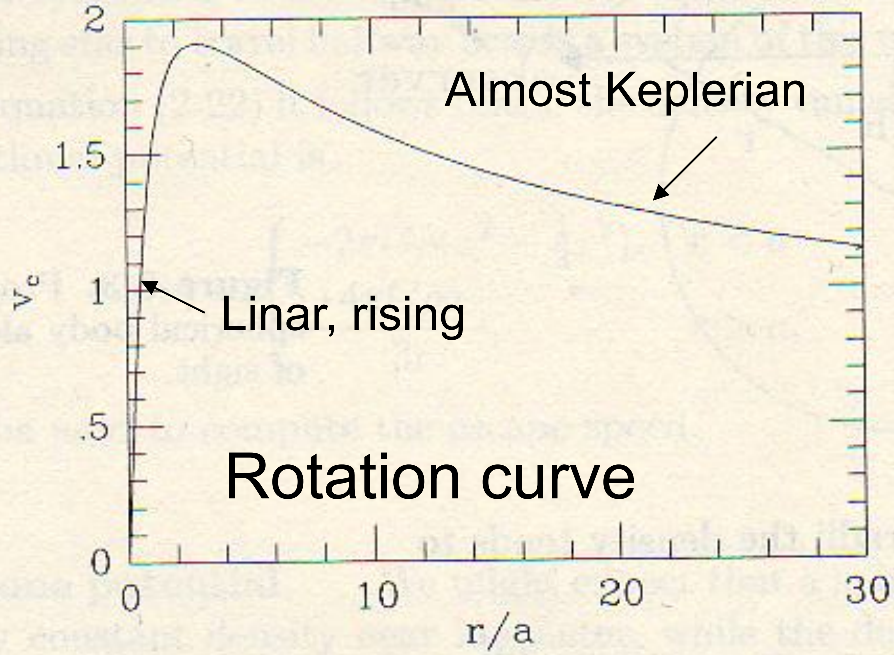



**Figure 2-4.** Circular speed versus radius for a body whose projected density follows the modified Hubble profile (2-39). The circular speed $v_c$ is plotted in units of $\sqrt{Gj_0\Upsilon a^2}$.

Thus $I_h(R) \propto R^{-2}$ at large $R$ and $I_h(R) \rightarrow constant$ as $R \rightarrow 0$, just as in the Hubble-Reynolds law (1-14).

Since the brightness distributions of many elliptical galaxies are fairly well fitted by the Hubble law, we conclude that the three-dimensional luminosity densities of elliptical galaxies cannot be very much like a point mass at large $r$. The circular speed is shown in Figure 2-4. It peaks at $r = 2.9a$ and then falls nearly as steeply as in the Keplerian case.

An important central-symmetric potential-density pair: singular isothermal sphere

**(e) Power-law density profile** Many galaxies have luminosity profiles that approximate a power law over a large range in radius. Consider the structure of a system whose mass density drops off as some power of the radius:

$$\rho(r) = \rho_0 \left(\frac{r_0}{r}\right)^\alpha . \tag{2-42}$$

The surface density of this system is

$$\Sigma(R) = \frac{\rho_0 r_0^\alpha}{R^{\alpha-1}} \frac{(-\frac{1}{2})!(\frac{\alpha-3}{2})!}{(\frac{\alpha-2}{2})!} . \tag{2-43}$$

Do you know why?

We assume that $\alpha < 3$, since only in this case is the mass interior to $r$ finite, namely

$$M(r) = \frac{4\pi\rho_0 r_0^\alpha}{3-\alpha} r^{(3-\alpha)} . \tag{2-44}$$

From equations (2-44) and (2-24) the circular speed is finite, namely

$$M(r) = \frac{4\pi\rho_0 r_0^\alpha}{3-\alpha} r^{(3-\alpha)}. \tag{2-44}$$

From equations (2-44) and (2-24) the circular speed is

$$v_c^2(r) = \frac{4\pi G\rho_0 r_0^\alpha}{3-\alpha} r^{(2-\alpha)}. \tag{2-45}$$

In Chapter 8 of MB we saw that the circular-speed curves of many galaxies are remarkably flat. Equation (2-45) suggests that the mass density in these galaxies may be proportional to $r^{-2}$. In Chapter 4 we shall find that this is the density profile characteristic of a self-consistent stellar-dynamical model called the **singular isothermal sphere**.

An empirical fact to which we'll return...

Equation (2-44) shows that $M(r)$ diverges at large $r$ for all $\alpha < 3$. However, when $\alpha > 2$, the potential difference in these models between radius $r$ and infinity, is finite. Thus the escape speed $v_e(r)$ from radius $r$ is given by

$$\begin{aligned} v_e^2(r) = 2\int_r^\infty \frac{GM(r')}{r'^2} dr' &= \frac{8\pi G\rho_0 r_0^\alpha}{(3-\alpha)(\alpha-2)} r^{(2-\alpha)} \\ &= 2\frac{v_c^2(r)}{\alpha-2} \qquad (\alpha > 2). \end{aligned} \tag{2-46}$$

Over the range $3 > \alpha > 2$, $(v_e/v_c)^2$ rises from the value 2 that is characteristic of a point mass, toward infinity. Since the light distributions of

# ASTB23 - Lecture L20
# Potential - density pairs (continued)

Flattened systems
- Plummer-Kuzmin
- multipole expansion & other transform methods

There is nothing more practical than theory:
- Gauss theorem in action
- using v = sqrt(GM/r)

Very frequently used: spherically symmetric Plummer pot. (Plummer sphere)

## 1 Plummer-Kuzmin Models

Consider the spherical potential

$$\Phi_P = -\frac{GM}{\sqrt{r^2+b^2}}. \tag{2-47a}$$

By direct differentiation we find

$$\nabla^2\Phi_P = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi_P}{dr}\right) = \frac{3GMb^2}{(r^2+b^2)^{5/2}}. \tag{2-48}$$

Thus from Poisson's equation we have that the density corresponding to the potential (2-47a) is

$$\rho_P(r) = \left(\frac{3M}{4\pi b^3}\right)\left(1+\frac{r^2}{b^2}\right)^{-\frac{5}{2}}. \tag{2-47b}$$

In 1911 H. C. Plummer used the potential-density pair that is described by equations (2-47) to fit observations of globular clusters. It is therefore known as **Plummer's model**. We shall encounter it again in §4.4.3(a) as a member of the family of stellar systems known as polytropes.

Now consider the axisymmetric potential

$$\Phi_K(R,z) = -\frac{GM}{\sqrt{R^2+(a+|z|)^2}}. \tag{2-49a}$$

Notice and remember how the div grad (nabla squared or Laplace operator in eq. 2-48) is expressed as two consecutive differentiations over radius! It's not just the second derivative.

Constant b is known as the core radius. Do you see that inside r=b rho becomes constant?

Often used because of an appealingly flat rotation curve v(R)--> const at R--> inf

## 2 Logarithmic Potentials

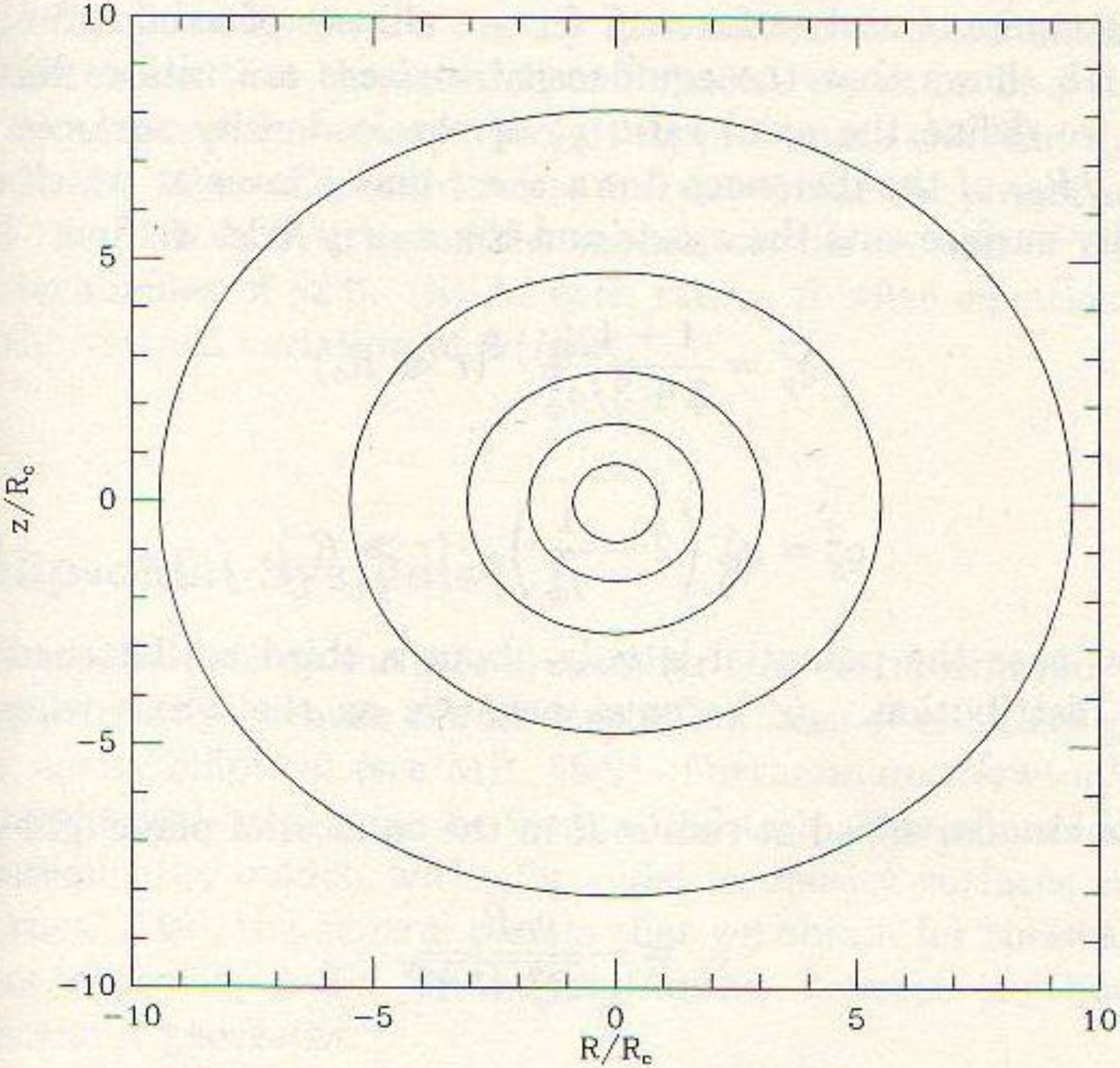
Since the Plummer-Kuzmin models have finite mass, the circular speed associated with these potentials falls off in Keplerian fashion $v_c \propto R^{-1/2}$ at large $R$. However, in §8-4 of MB it was shown that the rotation curves of spiral galaxies tend to be flat or rising at large radii. If at large $R$, $v_c \propto v_0$, a constant, then $d\Phi/dR \propto R^{-1}$, and hence $\Phi \propto v_0^2 \ln R + constant$ in this region. Therefore, consider the potential

$$\Phi_L = \tfrac{1}{2} v_0^2 \ln \left( R_c^2 + R^2 + \frac{z^2}{q_\Phi^2} \right) + \text{constant}, \qquad \text{(2-54a)}$$

where $R_c$ and $v_0$ are constants, and $q_\Phi \leq 1$. The density distribution to which $\Phi_L$ corresponds is

$$\rho_L(R,z) = \left( \frac{v_0^2}{4\pi G q_\Phi^2} \right) \frac{(2q_\Phi^2 + 1)R_c^2 + R^2 + 2(1 - \frac{1}{2} q_\Phi^{-2})z^2}{(R_c^2 + R^2 + z^2 q_\Phi^{-2})^2}. \qquad \text{(2-54b)}$$

At small $R$ and $z$, $\rho_L$ tends to the value $\rho_L(0,0) = (4\pi G)^{-1}(2 + q_\Phi^{-2})(v_0/R_c)^2$, and when $R$ or $|z|$ is large, $\rho_L$ falls off as $R^{-2}$ or $z^{-2}$.




Useful approx. to
galaxies if
flattening is sma

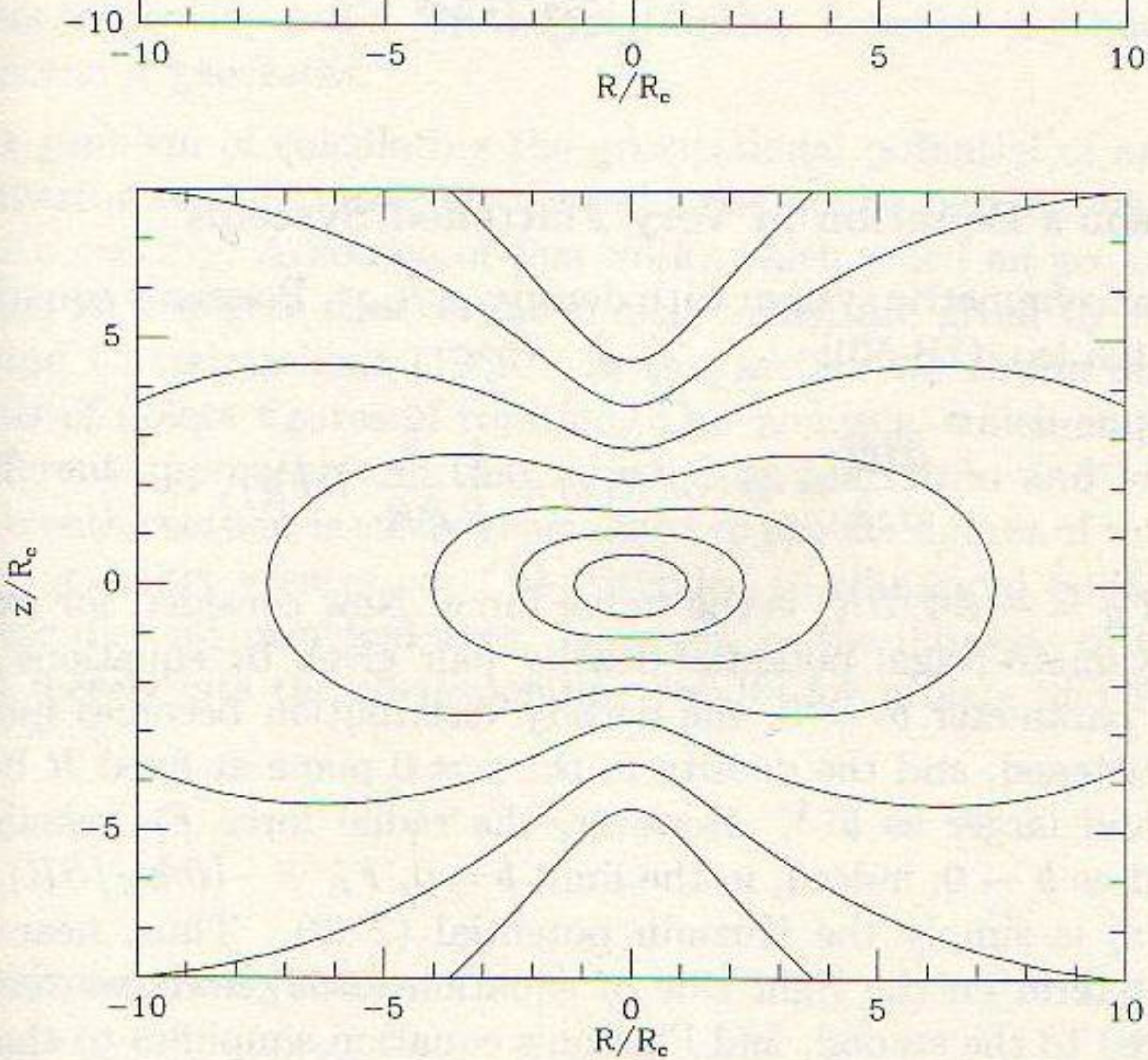



**Figure 2-8.** Contours of equal density in the $(R, z)$ plane for $\rho_L$ [eq. (2-54b)] when: $q_\Phi = 0.95$ (top); $q_\Phi = 0.7$ (bottom). In each case the contour levels are $0.1 v_0^2/(G R_c^2) \times (1, 0.3, 0.1, \ldots)$. When $q_\Phi = 0.7$ the density is negative near the $z$-axis for $|z| \gtrsim 7 R_c$.

Not very useful approx. to galaxies if flattening not <<1, i.e. q not close to

The equipotential surfaces of $\Phi_L$ are ellipses of axial ratio $q_\Phi$, but Figure 2-8 shows that the equidensity surfaces are rather flatter. In fact, if we define the axial ratio $q_\rho$ of the isodensity surfaces by the ratio $z_m/R_m$ of the distances down the $z$ and $R$ axes at which a given isodensity surface cuts the $z$ axis and the $x$ or $y$ axis, we find

$$q_\rho^2 = \frac{1+4q_\Phi^2}{2+3/q_\Phi^2} \quad (r \ll R_c) \tag{2-55a}$$

or

$$q_\rho^2 = q_\Phi^4 \left(2 - \frac{1}{q_\Phi^2}\right) \quad (r \gg R_c). \tag{2-55b}$$

In either case the potential is only about a third as flattened as the density distribution. $\rho_L$ becomes negative on the $z$-axis when $q_\Phi < 1/\sqrt{2}$.

The circular speed at radius $R$ in the equatorial plane of $\Phi_L$ is

(Log-potential)

$$v_c = \frac{v_0 R}{\sqrt{R_c^2 + R^2}}. \tag{2-54c}$$

The second part of the lecture is a repetition of the useful mathematical facts and the presentation of several problems

## Useful facts from Vector Calculus

$$\vec{\nabla} f = \hat{e}_x \frac{\partial f}{\partial x} + \hat{e}_y \frac{\partial f}{\partial y} + \hat{e}_z \frac{\partial f}{\partial z}$$

where $\hat{e}_x$ = unit vector in x-direct. (versor) etc.

in Cartesian coordinates $(x, y, z)$

$$\vec{\nabla} f = (\partial_x, \partial_y, \partial_z) f$$

⇐ shorter notation, $\partial_x := \frac{\partial}{\partial x}$

grad $f$ is a vector of derivatives (differential operators $\partial_i$)

---

Example:

$\Phi = \frac{1}{2}(x^2 + y^2 + z^2)\,\Omega^2$ is a potential ($\Omega$ = const.)

What is the increase of $\Phi$, $d\Phi$, when we move from point $(x_0, y_0, z_0)$ to $(x_0 + dx,\ y_0 + dy_0,\ z_0 + dz_0)$?

$d\Phi$ = x-slope · shift in x + y-slope · ...

$$= \left.\frac{\partial \Phi}{\partial x}\right|_0 \cdot dx + \left.\frac{\partial \Phi}{\partial y}\right|_0 \cdot dy + \left.\frac{\partial \Phi}{\partial z}\right|_0 \cdot dz$$

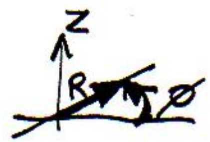
← this means "evaluated @ point "0"")

Shorter notation: $d\Phi = \left.\vec{\nabla}\Phi\right|_0 \cdot d\vec{r}$

where $\vec{r} = (x, y, z)$

## Gradient

In other coordinates:




($R \neq r$, $r$ = 3-D distance)

polar $\bar r = (R, \phi, z)$

$$\vec\nabla f = \hat e_r \frac{\partial f}{\partial r} + \frac{\hat e_\phi}{R}\frac{\partial f}{\partial \phi} + \hat e_z \frac{\partial f}{\partial z} \quad \text{or} \quad \vec\nabla f = \left(\partial_r, \frac{1}{R}\partial_\varphi, \partial_z\right) f$$

spherical $\bar r = (r, \theta, \varphi)$


$$\vec\nabla f = \left(\partial_r, \frac{1}{r}\partial_\theta, \frac{1}{r\sin\theta}\partial_\varphi\right) f$$

## Divergence

Cartesian

$$\vec\nabla \cdot \vec F = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = (\partial_x, \partial_y, \partial_z) \cdot \vec F$$

(You can only take divergence of a vector field.)

Polar (cylindrical) $\vec\nabla \cdot F = \frac{1}{R}\partial_R (R F_R) + \frac{1}{R}\partial_\phi F_\phi + \frac{\partial F_z}{\partial z}$

Spherical

$$\vec\nabla \cdot F = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 F_r + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta F_\theta) + \frac{1}{r\sin\theta}\partial_\phi F_\phi$$

**Laplacian = div grad** $\quad \vec{\nabla}^2 f = \nabla\cdot(\nabla f) = \text{div}\,(\text{grad} f)$

(You can only take Laplacian of a <u>scalar</u> field.)

$$\vec{\nabla}^2 f = (\partial_x, \partial_y, \partial_z)\cdot(\partial_x, \partial_y, \partial_z) f = (\partial_x^2 + \partial_y^2 + \partial_z^2) f$$

where $\partial_x^2 = \frac{\partial^2}{\partial x^2}$ etc.

Example: Let $f = \frac{1}{2}\Omega^2(x^2+y^2+z^2)$, then

$$\nabla^2 f = \nabla\cdot(\Omega^2 x, \Omega^2 y, \Omega^2 z) = \Omega^2 + \Omega^2 + \Omega^2 = 3\Omega^2$$

or

$$\nabla^2 f = (\partial_x^2 + \partial_y^2 + \partial_z^2) f = \frac{\partial^2}{\partial x^2} f + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \Omega^2 + \Omega^2 + \Omega^2 \quad \square$$

---

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \qquad \underline{\text{Cartesian}}$$

$$\nabla^2 f = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial f}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2} \qquad \underline{\text{cylindrical}}$$

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$

<u>(spherical)</u>

Significant simplification
in axisymmetric systems ($\partial_\phi \equiv 0$),
spherically-symmetric systems ($\partial_\theta \equiv 0$, $\partial_\phi = 0$)

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# Poisson equation and its solutions

$$\nabla^2 \Phi(\vec{r}) = 4\pi G \rho(\vec{r})$$

$$\rho = \frac{1}{4\pi G} \nabla^2 \Phi \quad \boxed{\Phi \to \rho}$$ , in general

$$\Phi = -4\pi G \left[ \frac{1}{r} \int_0^r \rho(x)\, x^2 dx + \int_r^\infty \rho(x)\, x\, dx \right]$$ in

spherically symmetric galaxies only (!)

Proof: ① $\nabla^2 \Phi = \frac{1}{r^2} \partial_r r^2 \partial_r \Phi$ (spherical symm.)

$$= (-4\pi G) \frac{1}{r^2} \partial_r r^2 \partial_r \left[ \frac{1}{r} \int_0^r (\dots) + \int_r^\infty (\dots) \right] =$$

= do the derivatives remembering that "r" is also found in integration limits, and that $\left\{ \partial_r \int_0^r f(x) dx = f(r) \right\} = \dots = 4\pi G \rho$ □

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② one can also simplify general solutions ...