Midterm exam ASTB23 2023. Stellar Astrophysics. Problem SOLUTIONS

Introduction: Cepheid pulsations

The Sun and other stars undergo oscillations, not unlike a large the air in a spherical subwoofer, where acoustic waves and modes are trapped inside. The wavelengths of essential, fundamental modes of lowest frequency, are on the order of their diameter (both in subwoofers and stars). The only difference is that waves in stars are subject to gravity forces, but we will neglect it here. Surface oscillations in case of variable stars called Cepheids cause observable brightness variations with easily measurable period Π.

The name Cepheids derives from constellation Cepheus in which the second such supergiant star was found (δ Cep, in year 1784). In the same year another star, η Aql, in constellation Aquila (The Eagle) was identified as a Cepheid.

1 [20 p.] First, derive the radii

Here are observations of the two stars (iaverage surface temp., average luminosity and the period of fundamental mode of oscillations):

δ Cep: $T_{eff} = 5790$ K, $L = 2000 L_\odot$, $\Pi = 5.36$ days

η Aql: $T_{eff} = 5100$ K, $L = 2630 L_\odot$, $\Pi = 9.84$ days

Derive the average radii R of the two stars in units of solar radius (you can use the fact that the sun has $T_{eff} = 5780$ K, and $L = 1 L_\odot$). Check the units, compute the values. Judge the plausibility of results.

2 [35 p.] Then derive the masses from the periods of oscillations

You know different ways in which the central temparature T_c of a star can calculated. Use any method you know (either virial theorem or the simplified solution of the hydrostatic structure equation, plus equation of ideal gas) to write a formula estimating T_c to the accuracy of at least an order of magnitude, and even a somewhat better accuracy, for a given mass M and radius R of the star.

The square of the soundspeed v_s inside a star is given by $v_s^2 = dP/d\rho$. If the gas is adiabatic, and the pressure and density variations in a wave are connected through polytropic index $\gamma = 5/3$, then the definition of v_s in the stellar core can be greatly simplified. Write the dependence of v_s on stellar mass and radius.

Astronomers observed that the most important modes of Cepheid oscillations have periods obeying the following empirical law

$$\Pi \approx 2.66 R/v_s.$$

Using the already derived formula for radius R, derive the formula for $M/M_\odot$. Check the units, evaluate the masses. If the result looks plusible, say why you think so.

Summarize your results in a small table listing the star's name, computed R and M in units of $R_\odot$ and $M_\odot$.

3 The possibly useful constants:

$G = 6.67259 \cdot 10^{-11}$ m^3 kg^{-1} s^{-2} = $6.67259 \cdot 10^{-8}$ cm^3 g^{-1} s^{-2} (gravity)
$k = 1.3807 \cdot 10^{-23}$ J/K = $1.3807 \cdot 10^{-16}$ erg/K (Boltzmann const.)
$m_H = 1.66054 \cdot 10^{-27}$ kg = $1.66054 \cdot 10^{-24}$ g (hydrogen mass)
$\sigma = 5.67051 \cdot 10^{-8}$ J m^{-2} s^{-1} K^{-4} = $5.67051 \cdot 10^{-5}$ erg m^{-2} s^{-1} K^{-4} (Stefan-Boltzmann)
$c = 299792458$ m/s $\simeq 3 \cdot 10^5$ km/s
$M_\odot = 1.9891 \cdot 10^{30}$ kg = $1.9891 \cdot 10^{33}$ g
$R_\odot = 6.9598 \cdot 10^8$ m = $6.9598 \cdot 10^{10}$ cm
1 AU = 150 mln km = $1.5 \cdot 10^{11}$ m; 1 pc = 206265 AU = $3.08568 \cdot 10^{16}$ m

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σT_{eff}^4 is the flux from unit area of the surface. Flux times the surface area of a star equals star's luminosity L. Therefore

$$R^2 = \frac{L}{4\pi\sigma T_{eff}^4}$$

Using this equation for the sun, then dividing equations for the star and the sun side by side, we have

$$R/R_\odot = (L/L_\odot)^{1/2}(T_{eff}/T_{\odot,eff})^{-2}.$$

Next, simplifying the equation $(dP/dr)/\rho = -GM(r)/r^2$ gives the following estimate: $P/\rho \approx GM/R$, consistent with a virial estimate of temperature since, according to the ideal gas equation, $P/\rho = kT/(\mu m_H)$. The soundspeed formula is therefore $v_s^2 = \gamma P/\rho \approx \gamma GM/R$. Units are ok; numerical values will be evaluated later (they are not yet needed, as we continue to manipulate symbolic equations.)

Squaring $\Pi \approx 2.66R/v_s$ and rearranging terms, we have

$$M \approx (2.66/\Pi)^2(R^3/\gamma G)$$

or

$$M/M_\odot \approx (2.66/\Pi)^2(R_\odot^3/\gamma G M_\odot)\,(R/R_\odot)^3$$

Units check out ok.

Converting all the quantities to SI units (e.g., Π to seconds), we can evaluate the masses in units of solar mass. The numerical results, including radii, are approximately
δ Cep: $R \approx 44.6R_\odot$, mass $M \approx 4.5M_\odot$
η Aql: $R \approx 65.9R_\odot$, mass $M \approx 4.3M_\odot$
Plausible? Yes. Stars much more luminous than the Sun must also be significantly more massive and have a larger radius. The mass range among stars is much narrower than the luminosity or temperature range, so a stellar mass of four or five $M_\odot$ seems quite plausible.

Wikipedia says:
δ Cep: $R \approx 44.5R_\odot$, mass $M \approx 4.5M_\odot$
η Aql: $R \approx 65.9R_\odot$, mass $M \approx 4.3M_\odot$
So we've got the right result.

Last name + student number: ....

Part II – TRUE-OR-FALSE test. Circle Y or N. If your answer is N, circle at least one wrong word or number for credit. Disregard grammatical errors and typos.

[Y **N**] Aristarchus of Samos **proved** that the sun and Moon are circling **the Earth** at distances he computed from the eclipses and quarter-Moon phase observations.

[**Y**] Passing through a slab characterized by opacity $\tau = 0.5$ the beam of radiation weakens to about 61% of the original flux.

[Y **N**] The Coulomb barrier of repulsion between two protons inside the sun is about **15 times** higher than the mean thermal energy of protons

[**Y** N] The pressure of nonrelativistic gas equals (2/3) of its energy density, and the pressure of gas of photons is equal to 1/3 of its energy density

[Y **N**] Combining the mass (or continuity) equation of stellar structure with the hydrostatic equation gives $dP = (1/4\pi)Gm\,dm/\mathbf{r^3}$, where $m = m(r)$ is the mass inside radius r.

[Y **N**] The central temperature of a sun-like star is of order 15 **thousand** Kelvin

[**Y** N] The three Kirchhoffs laws describe the laws of emission and absoption of gas either behind or in front of gas of different temprature.

[Y **N**] Inside a normal star, plasma in equilibrium has **the same** mean squared velocities of electrons and protons

[**Y** N] Stars on Main Sequence have the approximate scaling: $L \sim M^{3.5}$

[**Y**] If the sun were a burning chunk of coal, producing a constant luminosity of $1L_\odot$, it would burn completely in several thousand years

[Y **N**] The ideal gas law reads $P = \rho k_B T/(\mu m_H)$; symbol m_H stands for the **mean molecular mass** of the gas.

[**Y** N] The optical depth of the sun is of order 10^{11}, based on Thompson scattering assumption (free-free scattering).

[**Y**] If one star has twice the central temperature than another star but the same total mass, then it must be smaller or much more densely built near the center.

[Y **N**] Eddington confirmed the **special** theory of relativity and its chief achievement $E = mc^2$, by organizing observations of a solar eclipse on Principe Island in 1919.

[Y **N**] The energy of the slow **Kelvin-Helmholtz contraction** of the sun could provide the solar luminosity for its age, more than 4 Gyr.

[**Y** N] The limb darkening happens because we see down to a shallower layer of the star at the edge of the visible disk of the star

[Y **N**] Photoionization does **not** contribute to the opacity of gas, only the free-free and bound-free processes do so.

[**Y** N] $P = nkT$ is ideal gas law

[Y **N**] Einstein proved that **every known** elementary particle has a non-zero rest mass following from $E = mc^2$

[**Y** N] The main branch of p-p reaction chain producing 85 percent of luminosity of the sun happens in three steps, in which 4 protons gradually added to products of the previous steps, eventually turn into one alpha particle, two positrons and two electron neutrinos.

[**Y** N] One way to understand the quantum tunneling is to draw consequences from the quantum-mechanical uncertainty principle

[**Y** N] Unlike in Compton scattering, in Thomson scattering the wavelength shift of the photon is unobservably small