Posted: 19 Sep, due: 28 Sep.

ASTB23 (STARS AND PLANETS) PROBLEM SET #1.

Points in the square brackets give the relative weight with which the problems count toward the final score for the assignment. If you need any physical constants you may find them in the textbook or on internet. Always state and properly check the physical units in your formulae *before* plugging in constants and input data. Cf. methodology of solutions to problems in tutorial notes.

1 [25 p.] Wattage of the sun, and a thought experiment

(a) Derive the mass of the sun $M_\odot$ based on Earth's orbital motion around the sun. Assume trajectory to be circular. Set up an equation for the linear speed Earth's v_K (a.k.a. Keplerian speed), by equating the centripetal acceleration $-v_K^2/r$, where $r = 1$ AU = 149.5 mln km = 1 AU (as we derived in the tutorial), with the acceleration due to gravity of the sun. To stablish v_K, use the period of motion (take it to be $P = 1$ year) and the known orbital radius. Gravitational constant equals $G = 6.6743\text{e-}11$ m^3/(kg s^2). Compare the result with solar mass found in wikipedia.

(b) Based on the flux of radiation falling on Earth (1360..1370 W/m^2), in the tutorial we derived the luminosity (wattage) of the sun. Assume the number is $L_\odot = 3.85 \cdot 10^{26}$ W.

How many watts does 1 kg of the sun's interior produce, on average, from nuclear reactions? (Use $L_\odot$ and $M_\odot$.) Compare the sun with a car battery, which supplies power 2kW and weighs 10 kg. Which body is more efficient and by what factor (to within an order of magnitude, i.e. a round power of 10)?

(c) Estimate to within order of magnitude the working time of each object. Assume 2*current age of the sun, discharge time of battery follows from its capacity of 20 kWh (kilowatt-hours). Compare (in J/kg) the energy per unit mass emitted over the usable lifetime of the two objects. Compare the orders of magnitude.

Conduct a thought experiment, where you endow each kg of the sun with efficiency of energy release qual to the car battery (200 W). What would be the main-sequence lifetime of the sun? Would life on Earth as we know it exist then? Would it be possible anywhere in the Solar System?

Hint: Assume the total energy emitted over lifetime remains the same, no matter how fast it is released; it only depends on the initial amount of 'fuel'.

2 [15 p.] What is the surface temperature of the sun?

Base your calculation on sun's known luminosity $L_\odot$, and assume that the surface is at temperature T and emits as blackbody according to the law: $F = \sigma T^4$, where F is the flux from the surface measured in W/m^2, and $\sigma = 5.67037e-8$ W/m^2/K^4 is the Stefan-Boltzmann constant.

Hint: We know the mean distance to the sun (d = 1 AU = 149.5 mln km) and its mean angular diameter on the sky: 0.5332 degrees. Derive sun's physical radius R in km or m, do not quote it from internet.

3 [30 p.] Radius of a famous star

The star Beta Pictoris, a.k.a. β Pic is, as the second letter β of Greek alphabet reveals, the second-brightest star in the southern constellation of Painter's Easel (Pictor in Latin). It is famous for its dusty disk, which is an analog to the young solar system.

Find out how many times larger is the physical radius of that star (R_*) than the solar radius ($R_\odot$), knowing its surface temperature $T_{eff} = 8250$ K (estimated from the peak wavelength of spectrum), distance $d_* = 19.3$ pc (obtained from parallax), and its apparent magnitude equal to $m = 3.86$ mag (making it visible in good conditions but inconspicuous to the naked eye).

Adopt the apparent magnitude of the sun equal to $m_\odot = -26.73$ mag, and sun's effective temperature $T_\odot = 5780$ K.

Reminder: $m = -2.5 \log I/I_0$ is the magnitude of a star (unit is mag or magnitudo, essentially, like radian, it is 1). I is the observed brightness, which is simply the flux of radiation received on Earth. If m and I don't carry any subscripts (perhaps the name of the filter cutting out a part of the spectrum only), we mean that they are bolometric, which means describe a total over all wavelengths. For a zero-magnitude star like the nearby, fiducial star Vega, $I = I_0 = 2.188 \cdot 10^{-8}$ W/m^2, which, if ever needed, defines the constant I_0.

4 [30 p.] Linear density "star"

Consider a star with mass M and radius R. Hydrostatic equilibrium equation reads

$$dP/dr = -Gm(r)\rho(r)/r^2$$

where $m(r)$ is mass inside radius r and $\rho(r)$ spatial density of gas at radius r.

Warmup consideration:

We are going to consider a more realistic density profile next. But to see how the methods work, let's consider first a constant density "star". Divide the star mentally into shells of thickness dr and density $\rho(r)$, and assume $\rho(r) = \rho_c = const.$

Mass function $m(r)$ can be obtained by integration of a known density distribution $\rho(r)$ over radius:

$$m(r) = \int_0^r dm = 4\pi \int_0^r r^2 \rho(r)\, dr$$

(Under the integral sign a math purist would replace r with some other similarly named dummy variable. I do not.) In our constant density case $\rho(r) = \rho = const.$, integration gives

$$m(r) = (4\pi/3) r^3 \rho\,.$$

That's obvious to anyone who knows the volume of a sphere! (This is how you compute it using calculus).

We have $M = m(R) = (4\pi/3)R^3\rho_c$, from which a constant in the density law can be expressed through global stellar parameters M and R:

$$\rho_c = \frac{M}{(4\pi/3)R^3}$$

It's just the mass of a sphere divided by its volume, i.e. the mean density $<\rho>=$ Total Mass/Volume. This expression applies to all stars with given their M and R. Their central density may be much higher than mean density for realistic density profiles.

Pressure $P(r)$ can be obtained by integration of the hydrostatic equation with its minus sign in front, from a starting point to layer r. This starting point is actually $r = R$, the surface, because it's there that pressure starts at value zero and keeps building up as we go deeper. This is an approximation for sure, because in real stars neither pressure nor density fall to exactly zero at what we consider stellar surface (photosphere). But they're so small compared with their central and mean values that approximating them by zero at the surface should not worry us.

The central pressure, in a general case, is $P_c = P(0)$:

$$P(0) = G \int_0^R m(r)\rho(r) r^{-2}\, dr.$$

In the special case of constant density, the integral is very simple, and the result reduces to

$$P(0) = \frac{3}{8\pi}\frac{GM^2}{R^4}.$$

We've gone through the dimensions of central pressure in other estimates done in lectures, so we know that this formula has right units of force divided by area. Plugging in the M and R of the sun, we obtain 1.3e+14 Pa (Pa is S.I. unit of pressure, 1 Pa = 1 N/m^2).

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Your task is to take a somewhat more realistic (linear) density profile

$$\rho(r) = \rho_c(1-x)$$

where $x = r/R$, and R is star's radius, and *redo all the calculations*.

A. Start from $m(r)$. Find central density such that $m(R) = M$. Show that a spherical body with linear density falloff has a central density ρ_c that is 4 times larger than the mean density $< \rho >$.

B. Perform calculation of $P(r)$ function. Check the units. Find $P_c = P(0)$ for this linear density model. Compute the value assuming solar M and R.

C. Knowing the equation of state of ideal gas, combine P_c and ρ_c into an estimate of T_c. Check the units again. You may assume that mean molecular weight of solar-composition gas is $\mu = 1.2$. Evaluate the temperature assuming solar values for mass and radius. Compare the result with an estimate obtained in the textbook from the virial theorem. By what factor is it different from the virial estimate? Are your T_c and the texbook T exactly the same physical quantity?