

ASTB23 (STARS, GALAXIES) PROBLEM SET #2.

Points in the square brackets give the relative weight with which the problems count toward the final score. If you need any physical constants or stellar data such as solar mass, radius, and luminosity, you may find them in our textbook or on the web. If you have obtained a solution by a somewhat different path and you have done a good job describing how you reached your conclusions, in principle you should get a full credit or almost a full credit. Also, remember that some questions are estimate questions, where 10 percent difference with the solution below does not matter. If you think your solution was misunderstood, please talk to the lecturer.

NOTICE

There are 5 problems in this set. Solve any chosen 4 of them. Solutions of only the 4 first submitted problems will be graded.

CLARIFICATIONS

1. In problem 3 on Lane-Emden equation, you were asked to *demonstrate* that a given function is a solution of the equation. The revised text below additionally clarifies that you don't need to derive it from the equation. Demonstrate here means *verify* that it is a solution by direct substitution. (Thanks Karmanjot!)

2. In the IDL/GDL code (problem 5), there may be things unfamiliar or a bit misleading for Pythonians. One has already been spotted. (Thanks Shuvarthy!) The $a > b$ in IDL means $\max(a,b)$, not a relationship (logical) operator. See comments in prob. 5.

Send email if you have further questions.

1 [25.] Eddington luminosity in a cold stellar envelope of a supergiant star

Please read section on Eddington luminosity limit in our textbook (p. 60). The derivation is based on a constant Thomson scattering coefficient and mainly-hydrogen composition, $\kappa = \sigma_{Th}/m_H$, but it is applicable as well to the envelope of stars that have a higher opacity coefficient. Find the limiting luminosity of a red supergiant star with $\kappa = 120 \text{ cm}^2/\text{g}$ (due to molecules and even the condensed solid dust grains), and express it in units of solar luminosity, if $M = 3M_\odot$.

What happens when a supergiant exceeds this limiting L ?

2 [25p.] Polytropic gas laws of normal and ultrarelativistic gas

A gas law that connects pressure P and density ρ directly via an equation

$$P = K\rho^\gamma$$

is called a *polytropic* gas law. (A purists may say *barotropic*, most astrophysicists will say *polytropic* or *adiabatic*; it's almost the same thing.) K is a constant, and γ is a nondimensional constant known as adiabatic index.

Notice that the gas still independently obeys the ideal gas law variously called Clapeyron's or Boyle's gas law, from which its temperature T can be obtained if needed, for any ρ or P . T is not seen in the $P(\rho)$ formula but is not constant; as a matter of fact T changes with density as $T \sim P/\rho \sim \rho^{\gamma-1}$.

Adiabatic behavior is observed in a volume of gas that does not have external heat supply. Adiabatic laws do not exactly apply to the core of the sun, which is subject to nuclear heating. Nevertheless, adiabatic law is very important:

- (i) it applies approximately outside the energy-producing core,
- (ii) surprisingly accurate models of other stars (brown dwarfs, white dwarfs, neutron stars) and even super-jovian exoplanets, none of which produce energy in nuclear reactions, can be built using adiabatic relation $P \sim \rho^\gamma$ (one example is the subject of next problem),
- (iii) atmospheres tend to be approximately adiabatic,
- (iv) the relationship applies to soundwaves and density waves in the sun and stars, and even the air in your room – compression and decompression in a wave happen on a time scale shorter than the heat conduction time scale, making the behavior of gas locally adiabatic.

Prove that the normal nonrelativistic (monatomic neutral or ionized) gas, has adiabatic index $\gamma = 5/3$. In such a gas (as we already know) the pressure P equals $2/3$ of the kinetic (thermal) energy density. Let's call such internal energy of disordered microscopic motions U (symbol used in gas thermodynamics), then $P = (2/3)U/V$.

Also prove that in ultra-relativistic gas, where as we know $P = (1/3)U/V$, the adiabatic index equals $\gamma = 4/3$.

To prove this, consider gas of particles in a thermally insulated tube of constant cross section A , ending with a movable piston. The length of gas column is L , and can grow by a small amount dL in our thought experiment, which will change both pressure and density. The proof should utilize two fundamental conservation laws.

Firstly, even if the gas volume $V = AL$ and density ρ change, the mass of gas in the tube is constant.

Secondly, the 1st law of thermodynamics expresses conservation of energy, $dQ = dE + dW$. It says that the amount of heat supplied (zero in adiabatic gas!) equals the change of internal kinetic energy of gas plus the mechanical work dW done by the gas (also known as PdV).

3 [25p.] Lane-Emden equation

The hydrostatic equation of stellar structure reads

$$\frac{1}{\rho} \frac{dP}{dr} = -G \frac{\int_0^r 4\pi x^2 \rho(x) dx}{r^2}$$

(For extra clarity I made an explicit distinction between radius r and radius as an integration variable x .) The integral gives mass inside radius r .

Multiply the equation on both sides by r^2 . Differentiate over r , to obtain an equivalent second-order ODE. More than a century ago, Lane and then Emden successfully formulated that equation for gaseous objects satisfying polytropic equation of state

$$P = K \rho^{1+(1/n)}$$

For historical reasons, instead of γ we use another constant n , given by the equation $1 + (1/n) = \gamma$.

You could substitute the polytropic relationship, and after changing the dependent variable from ρ to a non-dimensional θ obeying $\rho = \rho_c \theta^n$, you would derive the so-called Lane-Emden equation valid for any n .

Alas, analytical solutions of Lane-Emden equation were only found for $n = 0, 1,$ and 5 . Fortunately, one of those n 's ($n = 1$) happens to beautifully approximate the equation of state and the structure of neutron stars, as well as some giant planets! So let us cut to the chase, so to say, and only consider $n = 1$ in this problem.

Assume that $n = 1$ (i.e., a polytropic gas with $\gamma = 2$). Simplify your 2nd order differential equation by the change of variable from ρ to a similar but non-dimensional variable $\theta = \rho/\rho_c$, where $\rho_c = \text{const.}$ is the central density of the star.

Derive the Lane-Emden equation for $n = 1$

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) = -\theta$$

where ξ is a rescaled radius: $\xi = r/\alpha$. (You may take a look at the general form of Lane-Emden equation on wikipedia, and be surprised how little it varies from this equation.)

What expression for α , hiding constants G, K and so on, did you obtain?

Finally, demonstrate that the following function is a solution of $n = 1$ Lane-Emden equation

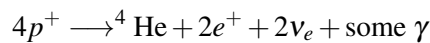
$$\theta(\xi) = \frac{\sin \xi}{\xi}$$

(you don't need to derive it from the equation, just verify that it is a solution).

Write the expression for $\rho(r)$ in a star obeying $P = K\rho^2$. What is the radius of such a star? Does it depend on ρ_c ? Does total mass of the star depend on central density? Formulate a conclusion about the R vs. M relationship in $n = 1$ polytropes.

4 [25p.] The p-p chain and its neutrinos

99% of energy production in the sun-like stars is from the so-called p-p chain thermonuclear reactions. Although the story how the chain works is a bit complicated, the input and output quantities are neatly summarized as



${}^4\text{He}$ is an alpha particle, or the nucleus of helium atom, consisting of 2 protons p^+ and two neutral but similarly massive neutrons. e^+ are the positrons, or anti-electrons (they later annihilate with surrounding electrons, releasing pure radiation in the form of γ rays).

This problem deals with the number of released very low-mass, weakly interacting particles called neutrinos (ν_e , subscript follows from their being of electron neutrino variety, two other types are also known).

On average, $E_\nu = 0.4$ MeV of energy (check on wikipedia what unit of energy is eV and how many eV are equal to 1 joule) is carried by **each** of the 2 neutrinos from the p-p chain reaction. In comparison, the total energy of about 27 MeV is released in other forms of radiation (mostly gamma) and eventually emitted from the surface of the star as degraded-energy, more numerous visible light photons.

Knowing the luminosity of the sun, estimate to within one or two accurate digits the number of p-p chain neutrinos, and the number per second of visible photons, leaving the sun. For a rough estimate, assume that visible photons all have energy $E_\gamma = hc/\lambda$ with λ in the middle of the wavelength range of the visible radiation. ($h = 6.626 \times 10^{-34}$ J s is the Planck's constant, and $c = 3 \times 10^8$ m/s is the speed of light in vacuum.)

Using this knowledge, calculate the number of neutrinos and photons passing through one cm^2 (area comparable with your eye) every second.

5 [25p.] Create your very own Sun (model)

Rewrite the program presented in the lecture and on the course page in Python, among others replacing all the graphics with Matplotlib graphics. Follow the same method as in the IDL script. For instance, copy exactly the somewhat arbitrary device to curtail the luminosity accumulating inside the star as it approaches the value of true solar luminosity; also follow the same prescription for opacity.

Perform the integration and see if you obtain the same(?) graphs of the non-dimensionalized pressure, density and temperature inside our nearest star.

What happens if you change the central temperature or pressure by 5%? What changes and by how many percent then?

CLARIFICATION for the code:

In IDL, the $>$ symbol is an arithmetic function, not a relationship (logical) operator. For example, $x > 5$ curiously means "maximum of x and 5". Similarly, $<$ means a minimum of what's on its two sides. In contrast, relational operators in IDL are: EQ, GT, LT, GE, LE, and mean $=, >, <, \geq, \leq$, or in Python: $==, >, <, >=, <=$. That tradition of using letters for relations comes from the ancient times (1970s) when programs were supposed to be written in literals, not symbols. Later that idea has died with the spread of C language, where $<$ means "less than" as God has intended. Also Fortran started accepting $<$ etc. as relational operators. But in IDL that could not be done. Someone already used the symbols for arithmetic functions, and unlike Python, but very much like Fortran, IDL wanted to be backward-compatible. This means (unlike Python2 vs. Python3) nobody is forced to rewrite their scripts after the release of a new version of the language.