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ASTB23 (Stars and Planets) Problem set #1. WITH SOLUTIONS

Points in the square brackets give the relative weight with which the problems count toward the final score for the assignment. If you need any physical constants you may find them in the textbook or on internet. Always state and properly check the physical units in your formulae *before* plugging in constants and input data. Cf. methodology of solutions to problems in tutorial notes.

1 [25 p.] Wattage of the sun, and a thought experiment

(a) Derive the mass of the sun $M_\odot$ based on Earth's orbital motion around the sun. Assume trajectory to be circular. Set up an equation for the linear speed Earth's v_K (a.k.a. Keplerian speed), by equating the centripetal acceleration $-v_K^2/r$, where $r = 1$ AU = 149.5 mln km = 1 AU (as we derived in the tutorial), with the acceleration due to gravity of the sun. To stablish v_K, use the period of motion (take it to be $P = 1$ year) and the known orbital radius. Gravitational constant equals $G = 6.6743\text{e-}11\ \mathrm{m}^3/(kgs^2)$. Compare the result with solar mass found in wikipedia.

SOLUTION

Acceleration balance gives

$$\frac{v_K^2}{r} = \frac{GM}{r^2}$$

which can be a bit simplified noticing r is on both sides. On the other hand we know that $v_K = 2\pi r/P$. Substitution yields a law known as a general form of Kepler's 3rd law around a point mass M:

$$P^2 = \frac{4\pi^2}{GM} r^3 ,$$

binding orbital period P to distance r (called semi-major axis) as $P^2 \sim r^3/M$. In our case,

$$M_\odot = \frac{4\pi^2 r^3}{GP^2} .$$

Don't forget to convert all units including P to S.I.! The result is $M_\odot = 1.99 \cdot 10^{30}$ kg.

(b) Based on the flux of radiation falling on Earth (1360..1370 W/m^2), in the tutorial we derived the luminosity (wattage) of the sun. Assume the number is $L_\odot = 3.85 \cdot 10^{26}$ W.

How many watts does 1 kg of the sun's interior produce, on average, from nuclear reactions? (Use $L_\odot$ and $M_\odot$.) Compare the sun with a car battery, which supplies power 2kW and weighs 10 kg. Which body is more efficient and by what factor (to within an order of magnitude, i.e. a round power of 10)?

SOLUTION

L/M = 3.85e26W/1.99e30kg $\sim$ 2e-4 W/kg, which is 6 orders of magnitude (million times!) smaller than battery's efficiency P/M = 2kW/10kg = 200 W/kg.

(c) Estimate to within order of magnitude the working time of each object. Assume 2*current age of the sun, discharge time of battery follows from its capacity of 20 kWh (kilowatt-hours). Compare (in

J/kg) the energy per unit mass emitted over the usable lifetime of the two objects. Compare the orders of magnitude.

Conduct a thought experiment, where you endow each kg of the sun with efficiency of energy release qual to the car battery (200 W). What would be the main-sequence lifetime of the sun? Would life on Earth as we know it exist then? Would it be possible anywhere in the Solar System?

Hint: Assume the total energy emitted over lifetime remains the same, no matter how fast it is released; it only depends on the initial amount of 'fuel'.

SOLUTION

Despite much lower specific power, battery cannot match the sun's longevity, 1e10 years. (Battery discharges over 20kWh/2kW = 10 hr, at most.)

A sun-like star over its lifetime emits of order 10 Gyr *(3e7 s/yr) *4e26 J/s   1.2e44 J = 1.2e38 MJ, or, on average per kg: 6e7 MJ/kg. Battery, not so much: 20 kWh = 20*3600 kJ = 72 MJ; which gives 7.2 MJ/kg over one recharge cycle.

Each kg of a star over its lifetime produces 7 orders of magnitude more total energy than the battery.

If our sun were emitting 200 W/kg like a battery, instead of 0.0002 W/kg, then its lifetime would be 6 orders of magnitude shorter than it is (given the same amount of energy emitted), i.e. not 1e10 yr but 1e4 yr (only 10 thousand yr). That's not enough for complicated life to emerge anywhere in the solar system, plus it surely is enough to evaporate the Earth at its present distance. True, if we were thousand times further from the sun, overheating would not be a problem, but complicated life would still not evolve enywhere in the Solar System in just 10^4 yrs.

2 [15 p.] What is the surface temperature of the sun?

Base your calculation on sun's known luminosity $L_\odot$, and assume that the surface is at temperature T and emits as blackbody according to the law: $F = \sigma T^4$, where F is the flux from the surface measured in W/m^2, and $\sigma = 5.67037e - 8$ W/m^2/K^4 is the Stefan-Boltzmann constant.

Hint: We know the mean distance to the sun (d = 1 AU = 149.5 mln km) and its mean angular diameter on the sky: 0.5332 degrees. Derive sun's physical radius R in km or m, do not quote it from internet.

SOLUTION

First, the radius of the sun R to its distance d is (*tan* of) half of the angular diameter, i.e. 0.5332/2 deg, or 0.004653 rad. Thus $R \simeq 0.004653d$ = 0.6956 mln km $\approx$ 0.7 mln km.

Next, since unit area of the sun emits flux $F = \sigma T^4$, and the area of the sun is $4\pi R^2$, then the total luminosity is the product of the two quantities: $L_\odot = 4\pi\sigma R^2 T^4$, from which

$$T = (L_\odot / 4\pi\sigma R^2)^{1/4} = 5780\text{K}.$$

(units agree as you can check).

Plausibility check: this T makes sense, because 'yellow' LED lights I buy at Home Depot have an advertised color temperature of 2800 to 3000 K, and daylight-simulating LEDs have, I believe, 4700 K (I never buy those so don't know precisely). The real sun seen from outside the Earth's atmosphere would appear even slightly bluer, so 5780 is quite plausible. Another plausibility check would be to use the so-called Wien's law to estimate that $T = 5780$K blackbody spectrum has a peak emission in the middle of the visible range, somewhere near 0.5μm wavelength.

3 [30 p.] Radius of a famous star

The star Beta Pictoris, a.k.a. β Pic is, as the second letter β of Greek alphabet reveals, the second-brightest star in the southern constellation of Painter's Easel (Pictor in Latin). It is famous for its dusty disk, which is an analog to the young solar system.

Find out how many times larger is the physical radius of that star (R_*) than the solar radius ($R_\odot$), knowing its surface temperature $T_{eff} = 8250$ K (estimated from the peak wavelength of spectrum), distance $d_* = 19.3$ pc (obtained from parallax), and its apparent magnitude equal to $m = 3.86$ mag (making it visible in good conditions but inconspicuous to the naked eye).

Adopt the apparent magnitude of the sun equal to $m_\odot = -26.73$ mag, and sun's effective temperature $T_\odot = 5780$ K.

Reminder: $m = -2.5 \log I/I_0$ is the magnitude of a star (unit is mag or magnitudo, essentially, like radian, it is 1). I is the observed brightness, which is simply the flux of radiation received on Earth. If m and I don't carry any subscripts (perhaps the name of the filted cutting out a part of the spectrum only), we mean that they are bolometric, which means describe a total over all wavelengths. For a zero-magnitude star like the nearby, fiducial star Vega, $I = I_0 = 2.188 \cdot 10^{-8}$ W/m^2, which, if ever needed, defines the constant I_0.

SOLUTION

For the star, $m_* = -2.5 \log I/I_0$, and for the sun, $m_\odot = -2.5 \log I/I_0$. Subtracting one from another, we obtain the ratio of observed fluxes.

$$10^{(m_* - m_\odot)/2.5} = I_*/I_\odot.$$

Writing out the fluxes as $I = L/(4\pi r^2)$ in each case, and substituting $L = 4\pi R^2 \sigma T_{eff}^2$ for each object, the ratio of observed fluxes becomes

$$I_*/I_\odot = (R_*/R_\odot)^2 (T_*/T_\odot)^4 (d_\odot/d_*)^2$$

where $d_\odot = 1$ AU is a known quantity (150 mln km).

Finally,

$$R_*/R_\odot = (T_*/T_\odot)^{-2} (d_*/d_\odot) 10^{0.2(m_* - m_\odot)}$$

. Units are nondim an both sides, ok. Evaluating the numbers brought to commoni S.I. system of units (I actually know that 1 pc = 206265 AU, so I didn't have to convert d_* into meters) one gets

$$R_*/R_\odot = (825/578)^{-2} (19.3 \cdot 206260) 10^{-30.59/5} = 1.48$$

Beta Pic is almost 1.5 times larger than the sun in radius. That is consistent with it being a hotter (compare T's), more massive, star.

Some of you chose to use the quoted value of the I_0 nomalization constant. That constant cancels out in the solution provided. But some of you used it and got a little too small radius, say, 1.32 times solar radius. It's ok, we won't subtract the points, if you promise in the future to stay clear of the use of constants that are not necessary for the solution. I think the quoted value (from internet) wasn't precisely the referring to the same range of wavelengths as the magnitues - that's why the difference poped out.

Finally, it is always good to comment on the accuracy of numbers we obtain (even though it's not part of the required solution of this particular problem – but sometimes you may beexplicitly asked to give a value and the standard error or a range of values). Here, we did have three accurate digits in all our input quantities, so we can reasonably quote the final result with 3 digits as well, although formally the accuracy with so many assumingly independent errors should sum up quadratically, so it'll be a few times 10^{-3} of the final answer. In any case, quoting 1.5 or conversely, many more digits after the decimal dot that pop out of your calculator, would be considered a mild mistake.

4 [30 p.] Linear density "star"

Consider a star with mass M and radius R. Hydrostatic equilibrium equation reads

$$dP/dr = -Gm(r)\rho(r)/r^2$$

where $m(r)$ is mass inside radius r and $\rho(r)$ spatial density of gas at radius r.

Warmup consideration:

We are going to consider a more realistic density profile next. But to see how the methods work, let's consider first a constant density "star". Divide the star mentally into shells of thickness dr and density $\rho(r)$, and assume $\rho(r) = \rho_c = const.$

Mass function $m(r)$ can be obtained by integration of a known density distribution $\rho(r)$ over radius:

$$m(r) = \int_0^r dm = 4\pi \int_0^r r^2 \rho(r)\,dr$$

(Under the integral sign a math purist would replace r with some other similarly named dummy variable. I do not.) In our constant density case $\rho(r) = \rho = const.$, integration gives

$$m(r) = (4\pi/3)r^3\rho\,.$$

That's obvious to anyone who knows the volume of a sphere! (This is how you compute it using calculus).

We have $M = m(R) = (4\pi/3)R^3\rho_c$, from which a constant in the density law can be expressed through global stellar parameters M and R:

$$\rho_c = \frac{M}{(4\pi/3)R^3}$$

It's just the mass of a sphere divided by its volume, i.e. the mean density $<\rho>=$ Total Mass/Volume. This expression applies to all stars with given their M and R. Their central density may be much higher than mean density for realistic density profiles.

Pressure $P(r)$ can be obtained by integration of the hydrostatic equation with its minus sign in front, from a starting point to layer r. This starting point is actually $r = R$, the surface, because it's there that pressure starts at value zero and keeps building up as we go deeper. This is an approximation for sure, because in real stars neither pressure nor density fall to exactly zero at what we consider stellar surface (photosphere). But they're so small compared with their central and mean values that approximating them by zero at the surface should not worry us.

The central pressure, in a general case, is $P_c = P(0)$:

$$P(0) = G\int_0^R m(r)\rho(r)r^{-2}\,dr.$$

In the special case of constant density, the integral is very simple, and the result reduces to

$$P(0) = \frac{3}{8\pi}\frac{GM^2}{R^4}.$$

We've gone through the dimensions of central pressure in other estimates done in lectures, so we know that this formula has right units of force divided by area. Plugging in the M and R of the sun, we obtain 1.3e+14 Pa (Pa is S.I. unit of pressure, 1 Pa = 1 N/m^2).

* * *

Your task is to take a somewhat more realistic (linear) density profile

$$\rho(r) = \rho_c(1-x)$$

where $x = r/R$, and R is star's radius, and *redo all the calculations*.

A. Start from $m(r)$. Find central density such that $m(R) = M$. Show that a spherical body with linear density falloff has a central density ρ_c that is 4 times larger than the mean density $<\rho>$.

B. Perform calculation of $P(r)$ function. Check the units. Find $P_c = P(0)$ for this linear density model. Compute the value assuming solar M and R.

C. Knowing the equation of state of ideal gas, combine P_c and ρ_c into an estimate of T_c. Check the units again. You may assume that mean molecular weight of solar-composition gas is $\mu = 1.2$. Evaluate the temperature assuming solar values for mass and radius. Compare the result with an estimate obtained in the textbook from the virial theorem. By what factor is it different from the virial estimate? Are your T_c and the texbook T exactly the same physical quantity?

SOLUTIONS:

A.

Using a formula for density $\rho = \rho_c(1-x)$, where ρ_c is the central density, and taking into account that $r = xR$, and $dr = R\,dx$, we get:

$$m(r) = 4\pi\rho_c \int_0^r (1-r/R)r^2 dr = 4\pi\rho_c R^3(x^3/3 - x^4/4)$$

Naming the total mass $m(R)$ as M, we find by substituting $x = 1$ that the central density

$$\rho_c = \frac{3M}{\pi R^3}$$

is exactly 4 times larger than the mean density $M/(4\pi R^3/3)$.

B.

$$P_c = P(0) = G\int_0^R m(r)\rho(r)r^{-2}\,dr = \frac{GM^2}{R^4}\int_0^1 (4x-3x^2)(1-x)\,dx = \frac{5\,GM^2}{12\,R^4}$$

This central pressure is $10\pi/9 \approx 3.5$ times larger than in the constant density model. The value is $P_c =$ 1.47e15 Pa.

C.

Knowing P_c and ρ_c, we can obtain the central temperature in our linear density model from ideal gas equation in the most useful form containing mass of hydrogen atom m_H and Boltzmann constant k: $P_c = \rho_c k T_c/(\mu m_H)$.

$$T_c = \frac{5\pi}{36}\frac{\mu m_H}{k}\frac{GM}{R}.$$

Units are ok [you needed to show it explicitly].

Substituting values ($\mu = 1.2$ applicable to solar composition, G =6.674e-11 S.I. units, k =1.3806e-23 J/K, m_H = 1.67e-27 kg) we get $T_c = 27.6$ million K, which is higher by a factor of 2.76 than the virial temperature 10 million K quoted in the texbook (p.18). That's OK, since the virial T is something a little different. It is an average temperature of a star, while we tried to estimate a higher, central, temperature.

The real Sun (and its more accurate models than our linear ansatz density), in turn, has $T_c = 15$ million K, a temperature that we overestimated by a factor of 1.8. See a numerical calculation of the structure of the sun on our course web page (to be discussed in one of the coming lectures). Even that numerical model is not super-exact, as it has $T_c = 17$ mln K, but it's much closer to 15 mln K than the linear density model considered here.

Acknowledgment: Many thanks to Karmanjot Sandh for noticing a mistake I made evaluating the front coefficient of the integral in pt. B. (This affected pt. C too.) Cookie points (activity points) to you!