Points in the square brackets give the relative weight with which the problems count toward the final score. If you need any physical constants or stellar data such as solar mass, radius, and luminosity, you may find them in our textbook or on the web. If you have obtained a solution by a somewhat different path and you have done a good job desribing how you reached your conclusions, in principle you should get a full credit or almost a full credit. Also, remember that some questions are estimate questions, where 10 percent difference with the solution below does not matter. If you think your solution was misunderstood, please talk to the lecturer.

1 [25.] Eddington luminosity in a cold stellar envelope of a supergiant star

Please read section on Eddington luminosity limit in our textbook (p. 60). The derivation is based on a constant Thomson scattering coefficient and mainly-hydrogen composition, $\kappa = \sigma_{Th}/m_H$, but it is applicable as well to the envelope of stars that have a higher opacity coefficient. Find the limiting luminosity of a red supergiant star with $\kappa = 120$ cm^2/g (due to molecules and even the condensed solid dust grains), and express it in units of solar luminosity, if $M = 3M_\odot$.

What happens when a supergiant exceeds this limiting L?

SOLUTION

$L_{Edd} = 4\pi GMm_Hc/\sigma_{Th} = 32000L_\odot(M/M_\odot)$ if opacity is assumed to be Thomson, $\kappa = \sigma_{Th}/m_H = 0.4$ cm^2/g. In our envelope opacity is, however, 120/0.4 = 300 times higher. Consequently, the Eddington luminosity is 300 times lower: $L_{Edd} = 107L_\odot(M/M_\odot) = 320L_\odot$. Once L exceeds this value, the envelope gets blown off. In reality, an oscillating envelope is shed nonuniformly in time, in the form of puffs of "smoke" (opacity is mostly due to solid grains of dust, like in a smoke).

2 [25p.] Polytropic gas laws of normal and ultrarelativistic gas

A gas law that connects pressure P and density ρ directly via an equation

$$P = K\rho^\gamma$$

is called a *polytropic* gas law. (A purists may say *barotropic*, most astrophysicists will say polytropic or *adiabatic*; it's almost the same thing.) K is a constant, and γ is a nondimensional constant known as adiabatic index.

Notice that the gas still independently obeys the ideal gas law variously called Clapeyron's or Boyle's gas law, from which its temperature T can be obtained if needed, for any ρ or P. T is not seen in the $P(\rho)$ formula but is not constant; as a matter of fact T changes with density as $T \sim P/\rho \sim \rho^{\gamma-1}$.

Adiabatic behavior is observed in a volume of gas that does not have external heat supply. Adiabatic laws do not exactly apply to the core of the sun, which is subject to nuclear heating. Neverthelass, adiabatic law is very important:

(i) it applies approximately outside the energy-producing core,

(ii) surprisingly accurate models of other stars (brown dwarfs, white dwarfs, neutron stars) and even super-jovian exoplanets, none of which produce energy in nuclear reactions, can be built using adiabatic relation $P \sim \rho^{\gamma}$ (one example is the subject of next problem),

(iii) atmospheres tend to be approximately adiabatic,

(iv) the relationship applies to soundwaves and density waves in the sun and stars, and even the air in your room – compression and decompression in a wave happen on a time scale shorter than the heat conduction time scale, making the behavior of gas locally adiabatic.

Prove that the normal nonrelativistic (monatomic neutral or ionized) gas, has adiabatic index $\gamma = 5/3$. In such a gas (as we already know) the pressure P equals 2/3 of the kinetic (thermal) energy density. Let's call such internal energy of disordered microscopic motions U (symbol used in gas thermodynamics), then $P = (2/3)U/V$.

Also prove that in ultra-relativistic gas, where as we know $P = (1/3)U/V$, the adiabatic index equals $\gamma = 4/3$.

To prove this, consider gas of particles in a thermally insulated tube of constant cross section A, ending with a movable piston. The length of gas column is L, and can grow by a small amount dL in our thought experiment, which will change both pressure and density. The proof should utilize two fundamental conservation laws.

Firstly, even if the gas volume $V = AL$ and density ρ change, the mass of gas in the tube is constant.

Secondly, the 1st law of thermodynamics expresses conservation of energy, $dQ = dE + dW$. It says that the amount of heat supplied (zero in adiabatic gas!) equals the change of internal kinetic energy of gas plus the mechanical work dW done by the gas (also known as $P\,dV$).

SOLUTION

From mass conservation, mass = $AL\rho = const$, which we may simply write as $L \sim \rho^{-1}$, as we just want to get the scaling laws, not the constant coefficients.

First law of thermodynamics says that $dU + dW = dU + PA\,dL = 0$, because: force * displacement dL = dW, and force = PA. In other words, $dU = -PA\,dL = -P\,dV$. From here on, you might have taken slightly different paths to combine the conservation laws, the result is of course the same.

For instance, I consider the increment dV while you may have used the equivalent $A\,dL$. $V = A\,L \sim L$, so in scaling laws L can replace V. Perhaps the shortest way is to express P as a known multiple of U right away.

We know that in two different cases of {non-relativistic, ultra-relativistic} gas, $P = \frac{\{2,1\}}{3}\frac{U}{V}$. Therefore

$$dU = -\frac{\{2,1\}}{3}\frac{U}{V}dV,$$
$$dU/U = -\frac{\{2,1\}}{3}\frac{dV}{V}$$

integration of which on both sides gives

$$\ln U = -\frac{\{2,1\}}{3}\ln V + const. = \ln V^{-\{2,1\}/3} + const.$$

(I mentioned in the lecture that the integration constant takes care of the problem of dimensional quantities being the argumant of logarithm. E.g., if prior to integration we wrote $dU/U = du/u$, where $u = U/U_0$ and U_0 is arbitrary dimensional constant, we'd get nondimensional versions of variables such as u as arguments of ln.)

$$U \sim V^{-\{2,1\}/3}$$

which together with $V \sim \rho^{-1}$ and $U = \frac{3}{\{2,1\}} PV \sim P/\rho$ gives

$$P \sim \rho^{1+\{2,1\}/3} = \rho^{\{5,4\}/3}.$$

We usually write this scaling as $P = K\rho^{\gamma}$ with $\gamma = 5/3$ in nonrelativistic vs. 4/3 in ultrarelativistic gas.

3 [25p.] Lane-Emden equation

The hydrostatic equation of stellar structure reads

$$\frac{1}{\rho}\frac{dP}{dr} = -G\frac{\int_0^r 4\pi x^2 \rho(x)\, dx}{r^2}$$

(For extra clarity I made an explicit distinction between radius r and radius as an integration variable x.) The integral gives mass inside radius r.

Multiply the equation on both sides by r^2. Differentiate over r, to obtain an equivalent second-order ODE. More than a century ago, Lane and then Emden successfully formulated that equation for gaseous objects satisfying polytropic equation of state

$$P = K\rho^{1+(1/n)}$$

For historical reasons, instead of γ we use another constant n, given by the equation $1 + (1/n) = \gamma$.

You could substitute the polytropic relationship, and after changing the dependent variable from ρ to a non-dimensional θ obeying $\rho = \rho_c \theta^n$, you would derive the so-called Lane-Emden equation valid for any n. Alas, analytical solutions of Lane-Emden equation were only found for $n = 0$, 1, and 5. Fortunately, one of those n's ($n = 1$) happens to beautifully approximate the equation of state and the structure of neutron stars, as well as some giant planets! So let us cut to the chase, so to say, and only consider $n = 1$ in this problem.

Assume that $n = 1$ (i.e., a polytropic gas with $\gamma = 2$). Simplify your 2nd order differential equation by the change of variable from ρ to a similar but non-dimensional variable $\theta = \rho/\rho_c$, where $\rho_c = const.$ is the central density of the star.

Derive the Lane-Emden equation for $n = 1$

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) = -\theta$$

where ξ is a rescaled radius: $\xi = r/\alpha$. (You may take a look at the general form of Lane-Emden equation on wikipedia, and be surprised how little it varies from this equation.)

What is the expression for α, hiding constants G, K and so on?

Finally, demonstrate that the following function is a solution of $n = 1$ Lane-Emden equation

$$\theta(\xi) = \frac{\sin \xi}{\xi}$$

Write the expression for $\rho(r)$ in a star obeying $P = K\rho^2$. What is the radius of such a star? Does it depend on ρ_c? Does total mass of the star depend on central density? Formulate a conclusion about the R vs. M relationship in $n = 1$ polytropes.

SOLUTION

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{1}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho$$

Substituting $P = K\rho^2$, we have $(1/\rho)dP/dr = 2K\,d\rho/dr$ (I hope you did not miss the inner derivative $d\rho/dr$!). Writing ρ as $\rho_c\theta$, we get

$$\frac{1}{r^2}\frac{d}{dr}r^2\frac{d\theta}{dr} = -\frac{2\pi G}{K}\,\theta,$$

which is the $n = 1$ Lane-Emden equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\xi^2\frac{d\theta}{d\xi} = -\theta$$

if we substitute $r = \alpha\xi$, where

$$\alpha^2 = K/(2\pi G).$$

The demonstration that $\theta = (\sin\xi)/\xi$ solved the differential equation is via simple differentiation and simplification: $\xi^2\frac{d\theta}{d\xi} = \xi\cos\xi - \sin\xi$, and another differentiation and multiplication of that yields

$$\xi^{-2}\frac{d}{d\xi}\xi^2\frac{d\theta}{d\xi} = \xi^{-2}\frac{d}{d\xi}\xi\cos\xi - \sin\xi = \xi^{-2}(\cos\xi - \xi\sin\xi - \cos\xi) = -\sin\xi/\xi = -\theta,$$

q.e.d. (Latin ”quid est demonstrandum” $\approx$ Eng. ”which was to be demonstrated”).

So the density is given by

$$\rho(r) = \rho_c\frac{\sin(r/\alpha)}{r/\alpha}$$

where $\alpha = [K/(2\pi G)]^{1/2}$. The sine function and the density hits zero value at such $r = R$ that $r/\alpha = \pi$, therefore

$$R = [\pi K/(2G)]^{1/2}.$$

The radius R does not depend on the central density, but the mass of a star does (it is simply proportional to ρ_c). This means that polytropes with $n = 1$ have a radius independent of mass.

4 [25p.] The p-p chain and its neutrinos

99% of energy production in the sun-like stars is from the so-called p-p chain thermonuclear reactions. Although the story how the chain works is a bit complicated, the input and output quantities are neatly summarized as

$$4p^+ \longrightarrow ^4\mathrm{He} + 2e^+ + 2\nu_e + \text{some }\gamma$$

^{4}He is an alpha particle, or the nucleus of helium atom, consisting of 2 protons p^+ and two neutral but similarly massive neutrons. e^+ are the positrons, or anti-electrons (they later anihilate with surrounding electrons, releasing pure radiation in the form if γ rays).

This problem deals with the number of released very low-mass, weakly interactig particles called neutrinos (ν_e, subscript follows from their being of electron neutrino variety, two other types are also known).

On average, $E_\nu = 0.4$ MeV of energy (check on wikipedia what unit of energy is eV and how many eV are equal to 1 jule) is carried by each of the 2 neutrinos from the p-p chain reaction. In comparison, the total energy of about 27 MeV is released in other forms of radiation (mostly gamma) and eventually emitted from the surface of the star as degraded-energy, more numerous visible light photons.

Knowing the luminosty of the sun, estimate to within one or two accurate digits the number of p-p chain neutrinos, and the number per second of visible photons, leaving the sun. For a rough estimate, assume that visible photons all have energy $E_\gamma = hc/\lambda$ with λ in the middle of the wavelength range of the visible radiation. (h = 6.626e-34 J s is the Planck's constant, and c = 3e8 m/s is the speed of light in vacuum.)

Using this knowledge, calculate the number of neutrinos and photons passing through one cm^2 (area comparable with your eye) every second.

SOLUTION

One reaction results in $2E_\nu = 0.8$ MeV in form of 2 neutrinos (0.4 MeV per neutrino), and $E = 27$ MeV in the form of visible photons. Each visible photon has according to quantum mechanics an energy of $h\nu$ (the frequency ν of visible photon equals $\nu = c/\lambda$ and can be computed assuming mean wavelength $\lambda \approx 0.5\mu$m).

Energy is radiated away from the sun in the ratio 0.8:27, or something like 1:30, in neutrino and photon forms. Assuming therefore that the luminosity of photons is (270/278)$L\odot$, while of neutrinos is only (8/278)$L\odot$, we have the following rates of radiation of these particles by the sun: $(270/278)L\odot/(hc/\lambda)$ (photons per second), and $(8/278)L\odot/E_\nu$ neutrinos/s.

Numerically, after converting MeV to J, we get these numbers released per second:

$$\dot{n}_\gamma \approx 10^{45}/s$$

and

$$\dot{n}_\nu \approx 3 \cdot 10^{38}/s$$

[NOTICE: There was a misprint in the original solution! Correct exponent 38 was mistakenly written as 36.]

Dividing by $4\pi(1AU)^2$ to obtain the flux at Earth, and multiplying it by 1 cm^2 area, we get the number of particles falling into your eye: $\sim 3 \cdot 10^{17}$ photons (1/3 billion billions) per second, and around 10^{11} or one hundred billion neutrinos per second. [Out textbook on page 155 makes a much bigger error than the misprint in our original solution, in stating "five million neutrinos pass through every cm^2 of the detector per second"!]

We don't see the less numerous neutrinos (they are also VERY weakly interacting with normal barionic matter of the eye, and even won't normally be stopped by the whole Earth), while the visible photons enter the eye in sufficient quantity to blind us temporarily.

5 [25p.] Create your very own Sun (model)

Rewrite the program presented in the lecture and on the course page in Python, among others replacing all the graphics with Matpliotlib graphics. Follow the same method as in the IDL script, for instance copy exactly the somewhat arbitrary device to curtail the luminosity accumulating inside the star as it approached value of true solar luminosity, as well as the same prescription of opacity.

Perform the integration and see if you obtain the same(?) graphs of the non-dimsionalized pressure, density and temperature inside our nearest star.

What happens if you change the central temperature or pressure by 5%? What changes and by how many percent then?

NOTICE

There are 5 problems in this set. Solve any chosen 4 of them. Solutions of only the 4 first submitted problems will be graded.