ASTB23 (STARS, GALAXIES) PROBLEM SET #3. SOLUTIONS

Points in the square brackets give the relative weight with which the problems count toward the final score. If you need any physical constants or stellar data such as solar mass, radius, and luminosity, you may find them in our textbook or on the web.

# 1 [40p.] Tidal force near a star - dangerous or not?

Tidal force is a differential force. Think of the famous problem of tides in Earth's oceans. The moon is pulling the central parts of Earth with some acceleration we can call mean acceleration. Ocean on the side closer to the Moon gets pulled stronger (since it's closer), while the opposite can be said of the oceans on the far side. The deviation from the mean acceleration is called tidal acceleration $a_{\rm tid}$, and gives rise to the tidal force that raises water level with respect to rocks that deform less willingly. Now forget the Moon – we will place ourselves near more massive objects.

Consider being close to the surface (which is at radius $R$ from the center of the object) of four different objects:

(1) Star Sirius A ($M = 2M_\odot$, $R = 1.71R_\odot$),

(2) White Dwarf, Sirius B ($M = 1M_\odot$, $R = 0.0084R_\odot$),

(3) Neutron Star PSR1257+12 ($M = 1.4M_\odot$, $R = 12$ km),

(4) Supermassive Black Hole (SMBH) in galaxy M87 ($M = 4\cdot 10^9 M_\odot$, $R = R_{\rm Sch}$ - we consider the SMBH's horizon radius, i.e. its Schwarzschild radius, to be its "surface").

$R_\odot$ =0.7e6 km is the solar radius, and $M_\odot$ =2e30 kg its mass. See tutorial notes from the latest tutorial for how to compute $R_{\rm Sch}$ in case (4).

**A.** You want to fly by each of the four objects in a spaceship, to a minimum distance $r = 3R$ in each case. Assume your full height is $h \approx 2$ m, and the diameter of a spaceship is 20 m ($h = 10$ m). Compute the tidal acceleration $a_{\rm tid}$. Decide which object: you or the ship is in more danger of disintegration.

Near which object will your life and/or the structural integrity of the spaceship be in peril? Assume that both you and your spaceship can survive differential acceleration up to 10g, where g is Earth's surface gravity acceleration $g = GM_E/R_E^2 = 9.81$ m/s$^2$.

**B.** A planet equal in mass and radius to Earth approaches the four objects' surfaces to within a distance such that 10 planetary radii separate the two surfaces. [Earth's radius is $R_E = 6371$ km, and its mass is $M_E = 3\cdot 10^{-6} M_\odot$.] Will the planet's gravity manage to hold it together or will it be ripped apart by tides around some object(s)?

SOLUTION

At a distance $r$ from mass $M$, tidal acceleration is equal

$$a_{tid} = \left| -\frac{GM}{(r-h)^2} + \frac{GM}{r^2} \right|$$

which we write as

$$a_{tid} = \frac{GM}{r^2}\left[\frac{1}{(1-h/r)^2} - 1\right].$$

Since $h/r \ll 1$, we can use the first two terms of Taylor expansion to approximate this expression as

$$a_{tid} \approx \frac{GM}{r^2}[2h/r] = 2\frac{GMh}{r^3}.$$

The units of $a_{tid}$ are the same as of $GM/r^2$, which are m/s$^2$ (ok).

**A.** All radii $R$ are given except for the SMBH: $R = R_{Sch} = 2GM/c^2$ =1.18e13 m. Now substitute $r = 3R$, and express $a_{tid}$ in units of $g = 9.81$ m/s$^2$

$$\frac{a_{tid}(r = 3R)}{g} \approx \frac{2GMh}{g(3R)^3}$$

Check the units: ok. Flyby is more dangerous for the rocket, because its $h$ is at least 5 times larger. We need to plug in the radius of the spaceship, $h = 10$ m, and see if we get more or lass than 10 (because the critical value is 10g).

(1) $\frac{a_{tid}}{g} \approx$ 1e-8 $\ll 10$
(2) $\frac{a_{tid}}{g} \approx$ 5e-2 $\ll 10$
(3) $\frac{a_{tid}}{g} \approx$ 8e+6 $\gg 10$ (!)
(4) $\frac{a_{tid}}{g} \approx$ 2e-11 $\ll 10$

The answer is somewhat counterintuitive. For all objects except NS, tidal acceleration is very small, flyby is very safe against tidal disruption. Even around the hugely massive SMBH (isn't that special). The only really dangerous objects are the very compact neutron stars! Extreme proximity to large point-mass source of gravity creates a very large gradient of force, which rips apart the spaceship (and you as well).

**B.** Here you may use the already derived formula with a modified $r = R + 10R_E + R_E = R + 11R_E$ (distance between centers). More elegantly, write explicitly $g = GM_E/R_E^2$ (surface gravity on Earth), $h = R_E$, and simplify the formula to

$$\frac{a_{tid}}{g} \approx 2\frac{M}{M_E}\frac{1}{(R/R_E + 11)^3}.$$

See if we get more or less than 1 (because the critical value is 1g, like on Earth):

(1) $\frac{a_{tid}}{g} \approx 0.17 < 1$
(2) $\frac{a_{tid}}{g} \approx 390 \gg 1$
(3) $\frac{a_{tid}}{g} \approx 700 \gg 1$
(4) $\frac{a_{tid}}{g} \approx 0.0004 \ll 1$

So the only tidally very SAFE approach an Earth-like planet can make to within its 10 radii from an object is around an SMBH. The second such case is Sirius A (pretty safe tidally), but it's a question if it can withstand the intense heating by Sirius A and not be evaporated. Compact stars will easily destroy the Earth tidally.

## 2 [20 p.] Energetics of falling into the black hole via a disk

A black hole has mass $M = 10^6 M_\odot$ and (Schwarzschild) radius $R_S = 2GM/c^2$. It is surrounded by a rotating disk of gas extending from the last stable orbit at $r = 3R_S$ to infinity. There is a small friction force (viscosity) acting withing the disk. The disk heats up and radiates because of this friction. Small disk elements of mass $m$

move on very tightly wrapped spiral paths (almost circles) toward the inner edge of the disk from infinity. At $r < 3R_S$, their orbits become unstable and the mass falls directly into the black hole without emitting further radiation.

Using Newtonian gravity potential $-GMm/r$ of mass $m$ in the field of mass $M$, and accounting for both the kinetic and potential energy of $m$ (assuming circular orbit at each time), find the energy of radiation $E_{rad}$ released by mass $m$ on its voyage from infinity to the black hole. What fraction of rest energy (i.e.$mc^2$) is $E_{rad}$? Compare your answer with the percentage of the rest mass in the process of fusion of hydrogen into helium theoretically releases (cf. Aston's results 1919, described in the textbook).

HINT: As an intermediate step, find the orbital speed $v$ as a function of radius $r$ from the balance of Newtonian gravity force $GMm/r^2$ and centrifugal force $mv^2/r$.

SOLUTION

Mechanical energy (potential plus kinetic) is equal

$$E = m(-GM/r + v^2/2)$$

Orbiting in a disk, $v^2$ of mass $m$ is given by the force balance (gravity vs. centrifugal) that reads $GMm/r^2 = mv^2/r$. Substituting $v^2 = GM/r$ into the energy equation, we get

$$E = -GMm/(2r)$$

anywhere in the disk. The mwchanical energy is not conserved, as it is gradually converted into radiation energy. Starting at radius infinity, it has value $E_\infty = 0$, and becomes more and more negative as the mass $m$ approaches the final orbital radius $r = 3R_S$, where the mechanical energy is equal

$$E_3 = -GMm/(6R_S)$$

The difference

$$E_{rad} = E_\infty - E_3 = GMm/(6R_S)$$

is radiated away. Since $R_S = 2GM/c^2$, we have $E_{rad} = mc^2/12$, or $E_{rad}/(mc^2) = 1/12$. A much larger fraction, about 8%, of rest energy gets radiated away than could be produced in thermonuclear reactions (between 0.7% and 0.8%). Throwing some matter (hydrogen, water, or banana peels) into accretion disk around a black hole, releases 10 times more energy than if this amount of matter exploded with the power of a hydrogen bomb.

## 3 [30 p.] Contraction and spin-up of a star to a neutron star

Suppose a stellar core comprising $M = 1.9$ solar masses contracts to form a neutron star (entirely composed of neutrons), while preserving the angular momentum of each gas particle.

A. [15p] Calculate the radius of the neutron star $R_{NS}$ and its mean density, assuming that neutrons in it are located on verices of a cubic grid and touching their neighbors. The radius of neutron equals $r_n$ = 0.87 fm. Mass of neutron is close to the mass of proton or hydrogen atom, $m_n = 1.6749\text{e-}27$ kg.

B. [15p] Compute the rotation period of a neutron star, if the initial rotation period was 19 days and initial core radius was 0.27 solar radii. Are there really neutron stars that rotate so fast?

SOLUTION

A. The cube has volume $(2r_n)^3$, and the neutron's mass is $m_n = 1.6749\text{e-}27$ kg, so the mean density of neutron matter is

$$\rho = m_n/(2r_n)^3 = 3.18\text{e}17\ \text{kg}/\text{m}^3$$

(318000000000000 times higher than water). Spherical core's mass $M = 1.9M_\odot$ equals $M = (4\pi/3)R_{NS}^3\rho$, therefore the radius of a neutron star is

$$(R_{NS}/r_n)^3 = (6/\pi)(M/m_n)$$

$$R_{NS} = r_n[(6/\pi)(M/m_n)]^{1/3} = 14.1\ \text{km}$$

B. Angular momentum of each mass element, locally preserved in the so-called homologous collase, is proportional to $\Omega R^2 = const.$, therefore the period scales like $P \sim 1/\Omega \sim R^2$. Thus the final period is $[R_{NS}/(0.27R_\odot)]^2$ times the initial period, or numerically: $P = 9.1$ ms.

Yes, there are NS that rotate that fast (110 times per second) and even somewhat faster. They are called millisecond pulsars.