ASTB23 (STARS & GALAXIES) PROBLEM SET #4. SOLUTIONS

Points in the square brackets give the relative weight with which the problems count toward the final score. If you need any physical constants, you may find them in the textbook(s) or on the web. Sorry for ugly typography of letter 'vee' as v which looks like Grek 'nu' (ν)

1 [26p.] Milky Way as a Logarithmic potential galaxy

Observations of the Milky Way galaxy reveal that the circular speed at the distance $R = 8.5$ kpc is 220 km/s, while it is 170 km/s at the distance of 2.2 kpc from the center.

Find two constant parameters R_0 and v_0, in units of kpc and km/s correspondingly, of the following logarithmic potential

$$\Phi(R,z) = \frac{v_0^2}{2} \ln \frac{R^2 + R_0^2 + z^2/0.9^2}{R_0^2},$$

that provides the best fit to the observational data.

What circular speed is found at infinity (i.e. when $R \gg R_0$)? What speed is found at $R = 0$?

Notice: The circular speed is defined in the midplane of the galaxy only, i.e. at $z = 0$.

SOLUTION

Evaluating the general expression for v_c^2 applied to the given i potential at $z = 0$ yields

$$v_c^2 = R\partial_R\Phi = \frac{v_0^2}{1 + R_0^2/R^2},$$

at the two given points where R and v_c are both known, we obtain two equations for two unknowns R_0 and v_0. Solving them yields

$$R_0^2 = \frac{1-(v_1/v_2)^2}{(v_1/v_2)^2 R_1^{-2} - R_2^{-2}} = \frac{(v_2/v_1)^2 - 1}{R_1^{-2} - (v_2/v_1)^2 R_2^{-2}}$$

(two equivalent forms), and

$$v_0 = v_i\sqrt{1 + R_0^2/R_i^2}$$

where you can choose $i = 1$ or 2. Units are evidently ok.

We have

$$R_0^2 = \frac{(220/170)^2 - 1}{8.5^{-2} - (220/170)^2 2.2^{-2}} (\mathrm{kpc})^2,$$

and

$$v_0 = 170\sqrt{1 + (R_0/2.2\mathrm{kpc})^2}\,\mathrm{km/s}.$$

Calculation gives $R_0 = 1.93$ kpc, and $v_0 = 226$ km/s. It is always good to quote an answer to only as many accurate digits as there were present in the input data, in this case three or less.

The asymptotic circular speed at infinity is $v_c(\infty) = v_0 = 226$ km/s. At $R = 0$ the speed is zero.

2 [24p.] Epicyclic and orbital frequency

Orbital frequency is $\Omega(R) = v_c(R)/R$, also known as angular frequency or angular speed (in radians per second) of circular orbital motion.

Epicyclic frequency is the frequency of radial oscillation of distance r around the guiding center (imaginary circular orbit, around which a real trajectory, e.g. of the sun, oscillates, between some minimum and maximum distance from the center of the galaxy). One can prove (but you do not have to!) that this frequency equals

$$\kappa^2(R) = R^{-3}\frac{d(R^4\Omega^2)}{dR}$$

in arbitrary potential, that is in arbitrary rotation curve.

Based on the formulae from the previous problem, show that the Galaxy near its center ($R \ll R_0$) has a linearly rising rotation curve $v_c(R)$, and an approximately constant velocity curve $v_c \approx const.$ at very large radii $R \gg R_0$. Compute the ratio of epicyclic to orbital frequency κ/Ω in these two asymptotic cases. (You don't have to use the precise $\Omega(R)$ curve, you may use two separte asymptotic expressions). Frequency is inversely proportional to period. Based on this, qualitatively describe the consequences of your results for the shape of the orbit of a star (like the sun) on a nearly circular orbit, in particular whether the orbit is closed. If not, then which way does it precesses in space. Make a qualitative top-view sketch of non-circular sun's trajectory.

SOLUTION

From speeds 220 km/s at 8.5 kpc and only a little more (226 km/s) at infinity, it is clear that beyond 8.5 kpc the rotation speed is almost flat (constant). In a flat rotation curve disk, $\kappa = \sqrt{2}\Omega$ by direct substitution of $\Omega = const./R$ into the formula for κ^2 given in the text.

At the other extreme of small radii, $R \ll R_0$, we find

$$v_c^2 = \frac{v_0^2}{1 + R_0^2/R^2} \approx \frac{v_0^2 R^2}{R_0^2},$$

$$v_c \approx (v_0/R_0)R,$$

a linear function of distance R. In that region, $\Omega = v_c/R = const$ (region rotates as a solid body). $\kappa = 2\Omega$.

Since, in fact, everywhere $\kappa > \Omega$, it takes less time for a star on a nearly circular orbit to execute one radial oscillation period than to circle the center of the galaxy once. The orbit turns in space in the the direction opposite to the to particle/star motion.

In addition, the ratio of periods will typically be irrational (except in the solid body rotation region, cf. below). The orbit drawn in inertial frame never closes on itself. It is a rosette (a turning ellipse) which in time fills the space inside a ring.

One can even predict precisely how far (in angle) the next pericenter (or apocenter) of the orbit is, compared with a given starting pericenter. That goes beyond the qualitative analysis, but for the record: difference of angular speeds $\kappa - \Omega$ is the rate of rotation of the underlying ellipse (precession rate). Multiplied by the epicyclic period, we get the azimuthal angular spacing between consecutive points of minimum or maximum distance from the center (pericenter/apocenter). It is $2\pi(1 - 1/\sqrt{2}) = \pi(2 - \sqrt{2})$ radians = 1.840 rad = 105.4 degrees.

ADDITIONAL COMMENT: There are only two force laws and rotation curve cases, in which an orbit closes on itself and does not precess. This is stated by the so-called Bertrand's theorem. One is when $\kappa = \Omega$ and that's around a point mass (elliptic trajectory, mass in the focus of ellipse). The other is Hooke's law $F \sim -R$ that produces $v_c \sim R$ (solid body rotation), where the precession is exactly 180 degrees in one radial oscillation period, such that the orbit is an ellipse centered on its center of symmetry (center of mass).

3 [23p.] Surface brightness in a disk galaxy NGC3011

Galaxy NGC3011 has an exponential disk dominating the visible light image. Surface brightness (luminosity per unit area) is given by the exponential law

$$I(R) = I_0 \exp(-R/R_d)$$

where I_0 is the central surface brightness, R distance from the center, and $R_d = 4$ kpc the exponential radial scale of the disk (e-folding distance). The total luminosity of the galaxy equals $L = 2.5 \cdot 10^{10} L_\odot$.

A. Considering that the total luminosity is the surface brightness $I(R)$ integrated over the *area* of the whole disk from $R = 0$ to $R = \infty$ (not over the radial distance, a mistake some students make!), compute I_0 in units of $L_\odot/\text{pc}^2$, solar luminosities per square parsec.

SOLUTION

$$L = 2\pi \int_0^\infty I(R)\, R\, dR = 2\pi I_0 R_d^2 \int_0^\infty x\, e^{-x}\, dx = 2\pi I_0 R_d^2 .$$

$$I_0 = L/(2\pi R_d^2) = 2.5 \cdot 10^{10}/(2\pi 4000^2)\, L_\odot/\text{pc}^2 \simeq 250\, L_\odot/\text{pc}^2.$$

B. What is the surface brightness of the galaxy at radius 8 kpc, $I(8kpc)$? Assuming a standard disk light-to-mass ratio $\Upsilon = 4M_\odot/L_\odot$ appropriate to many disk galaxies, convert your answer into the surface density of disk stars in units of $M_\odot$/pc^2 and compare with the knowledge of the solar neighborhood of our Milky Way (roughly 60 $M_\odot$/pc^2 of luminous matter).

SOLUTION

At 8 kpc the surface brightness amounts to $250\, e^{-8/4} \simeq 34 L_\odot/\text{pc}^2$. If $\Upsilon = 4M_\odot/L_\odot$ then $\Sigma(R = 8\text{kpc}) = 135\, M_\odot/\text{pc}^2$. This is more than the Milky Way's surface density by about a factor of two.

4 [27p.] Cosmological constant and the accelerated expansion of the Universe

Expansion of the Universe does not have one central point. Any point in space can be taken as a point from which the rest of space is expanding. Let's pick one such a point and draw a sphere of radius R around it. In an expanding universe, $R(t)$ is an increasing function of time t, while the amount of matter M inside the sphere of radius $R(t)$ remains constant.

The expansion of the universe can be approximated by the following dynamical equation

$$\ddot{R} = -GM/R^2 + (\Lambda/3)R$$

where $M = const.$ and each dot stands for $\frac{d}{dt}$ operation. The underlying assumptions will become clearer at the end of the course when we discuss flat spacetime and cosmological constant Λ. The first term is, as you see, the Newtonian gravity term, the second was introduced by Albert Einstein as sort-of antigravity opposing the gravitational attraction (In 1999 Λ was observationally discovered to be a small positive number; its unit is s^{-2}). This means that vacuum for some reason has nonzero energy per unit volume and likes to expand; this energy is called dark energy.

• Derive the law of the speed of universal expansion: $\dot{R}(t)$ and make a sketch of its time dependence. Based on that also provide a rough sketch of $R(t)$.

Hint: Define the potential function $\Psi(R)$ via:
$-d\Psi/dR = -GM/R^2 + (\Lambda/3)R$. • Find function $\Psi(R)$.

• Verify by taking the full time derivative of E that the dynamical equation has the following energy-like constant

$$E = \frac{1}{2}\dot{R}^2 + \Psi = const.$$

• Show that the universal expansion is initially dominated by the gravity term, but after a certain time the speed $\dot{R}$ stops decreasing and then increases, driven by the lambda-term, asymptotically leading to an exponential growth of the scale factor R.

[Hint: Cosmologists argue that $E = 0$ is the correct value of the constant. Follow their advice.]

• What is the e-folding timescale in the late stage?

[Zoom in on the text of this problem, R and $\dot{R}$ are sometimes hard to distinguish. If in doubt, as always, check the units!]

SOLUTION

$$\Psi(R) = -\frac{GM}{R} - \frac{\Lambda}{6}R^2$$

by single integration over R. We now show that

$$E = \frac{1}{2}\dot{R}^2 - \frac{GM}{R} - \frac{\Lambda}{6}R^2 = const.$$

This is because the full time derivative dE/dt vanishes:

$$\frac{dE}{dt} = \left[\ddot{R} + \frac{GM}{R^2} - \frac{\Lambda}{3}R\right]\dot{R} = 0 \ \text{(q.e.d.)}$$

Rewrite the definition of $E = 0$ equation as a formula for $\dot{R}$:

$$\dot{R} = \left[2GM/R + (\Lambda/3)R^2\right]^{1/2}$$

At small t, R is small but growing very fast. We have $\frac{2GM}{R} \gg \frac{\Lambda}{3}R^2$. Gravity initially dominates the Λ-term, which can be neglected. This simplifies the evolution equation to $\dot{R} = [2GM/R]^{1/2}$, which by the method of separation of variables has a solution that goes like $R(t) \sim t^{2/3}$ (check it!). The speed of expansion gradually decreases. Physically this is because the dominant gravity is an attractive force opposing expansion, so it should slow it.

The plot of $\dot{R}$ has a single positive minimum [at $R = (3GM/\Lambda)^{1/3}$].

Conversely, at sufficiently large t and R, we have $\frac{2GM}{R} \ll \frac{\Lambda}{3}R^2$, and we can neglect the gravity term. Dark energy dominates, and not only the radius R but also its rate of change $\dot{R}$ grow. The expansion becomes exponential:

$$\dot{R} \approx \sqrt{\Lambda/3}\,R$$

$$dR/R \approx \sqrt{\Lambda/3}\,dt$$

$$\int dR/R \approx \sqrt{\Lambda/3}\int dt$$

$$\ln R(t) + const \approx \sqrt{\Lambda/3}\ t$$

$$R(t) \approx R_0 \exp[\sqrt{\Lambda/3}\,t]$$

The exponential growth has timescale of e-folding equal $\sqrt{3/\Lambda}$. (Timescale's unit is seconds, because unit of Λ is s^{-2}.)