Elements of Celestial Mechanics

Perturbation theory of orbits:

L7 General (analytical) perturbations

- relativistic precession, solar sail

Special (numerical) perturbation theory

- Euler and RK methods of integration

L8 Energy integral methods

3 Body Problem

Chaos in the solar system

L7
General theory of perturbations (analytical)

Joseph-Louis
Lagrange (1736-1823)

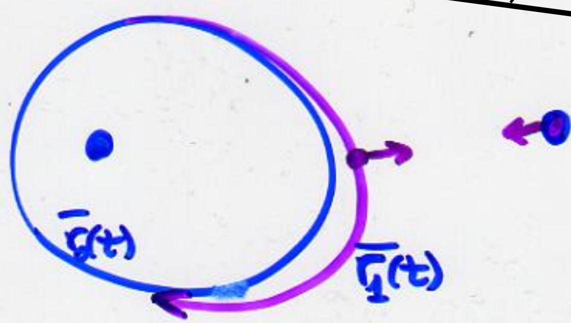
1st order perturbation theories

$$\vec{r}(t) = \underbrace{\vec{r}_0(t)}_{\text{unperturbed}} + \underbrace{\vec{r}_1(t)}_{\text{1st order}} + \ldots$$

$$\ddot{\vec{r}} = \vec{F}(\vec{r})$$ expanded as

$$\ddot{\vec{r}}_0 + \ddot{\vec{r}}_1 + \ldots = \vec{F}(\vec{r}_0 + \vec{r}_1 + \ldots) = \vec{F}_0 + \vec{F}_1 + \ldots = \vec{F}(\vec{r}_0) + \nabla\vec{F}(\vec{r}_0)\cdot\vec{r}_1 + \ldots$$

→ 1st order perturbation equations



philosophy:

perturbation (variables with index 1) are evaluated along the *unperturbed* trajectory (index 0)

Carl F. Gauss used the radial (R) and transversal (T) components of perturbing forces (accelerations) to compute torque (r T) and the orbital energy drain/gain rate (dE/dt = force * dr/dt) to find

$$\dot{L}, \dot{E} \longrightarrow \dot{a}, \dot{e}, \dot{\omega}$$

along the unperturbed orbit

Calculate $\dot{E}, \dot{\ell} \longrightarrow \dot{a}, \dot{e}$ on the unperturbed orbit

ellipse

$$\dot{E} = \vec{v} \cdot F \qquad dE = d\vec{r} \cdot \vec{F}$$

$$\dot{\ell} = |\vec{r} \times F|$$

$$\frac{dE}{dt} = -\frac{GM}{2} \frac{d}{dt}\left(\frac{1}{a}\right) = \frac{GM}{2a^2} \frac{da}{dt} \Rightarrow$$

$$\Rightarrow \quad \dot{a} = \dot{E} \frac{2a^2}{GM} = \vec{v} \cdot \vec{f} \, \frac{2a^2}{GM} =$$

$$= \frac{2}{n\sqrt{1-e^2}} \left[eR \sin\theta + T(1+e\cos\theta) \right]$$

(θ = true anomaly)

$\left(n = \sqrt{\frac{GM}{a^3}} \right)$

$$\vec{F} = (R, T)$$

n = 'mean motion' = mean angular speed, often denoted as Ω in other contexts

(R, T) = time-dependent components of perturbing force (acceleration)

Gauss Theory of Perturbations

Perturbing force (acceler.) $\vec{F} = (R, T)$

R ← radial, T ← transverse

Momentary rate of change of

a = semi-major axis

e = eccentricity

$\varpi = \omega + \Omega$ (modified) longitude of pericenter

[Note: $n = \sqrt{GM/a^3}$ = mean motion]

θ = true anomaly

$$\dot{a} = \frac{2}{n\sqrt{1-e^2}}\left[eR\sin\theta + (1+e\cos\theta)\,T \right]$$

(copying from the previous page)

$$\dot{e} = \frac{\sqrt{1-e^2}}{na}\left[R\sin\theta + \left(\cos\theta + \frac{e+\cos\theta}{1+e\cos\theta}\right)T\right]$$

$$\dot{\omega} = -\frac{\sqrt{1-e^2}}{nae}\left[R\cos\theta - \sin\theta\left(\frac{2+e\cos\theta}{1+e\cos\theta}\right)T\right]$$

(Derived like da/dt, from energy and angular momentum change)

If you need to average these rates over time (one orbital period), use angular momentum constant

$$l = r^2\frac{d\theta}{dt} = \sqrt{GMa(1-e^2)} = na^2\sqrt{1-e^2}$$

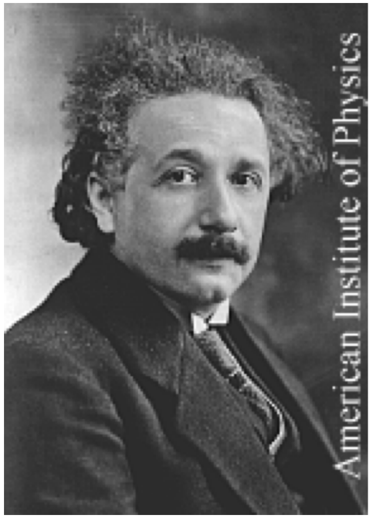
to change $\int_0^P (\ldots)\,dt = \int_0^{2\pi} (\ldots)\frac{r^2}{na^2\sqrt{1-e^2}}\,d\theta$

change variables and use the equation of ellipse for r(θ):

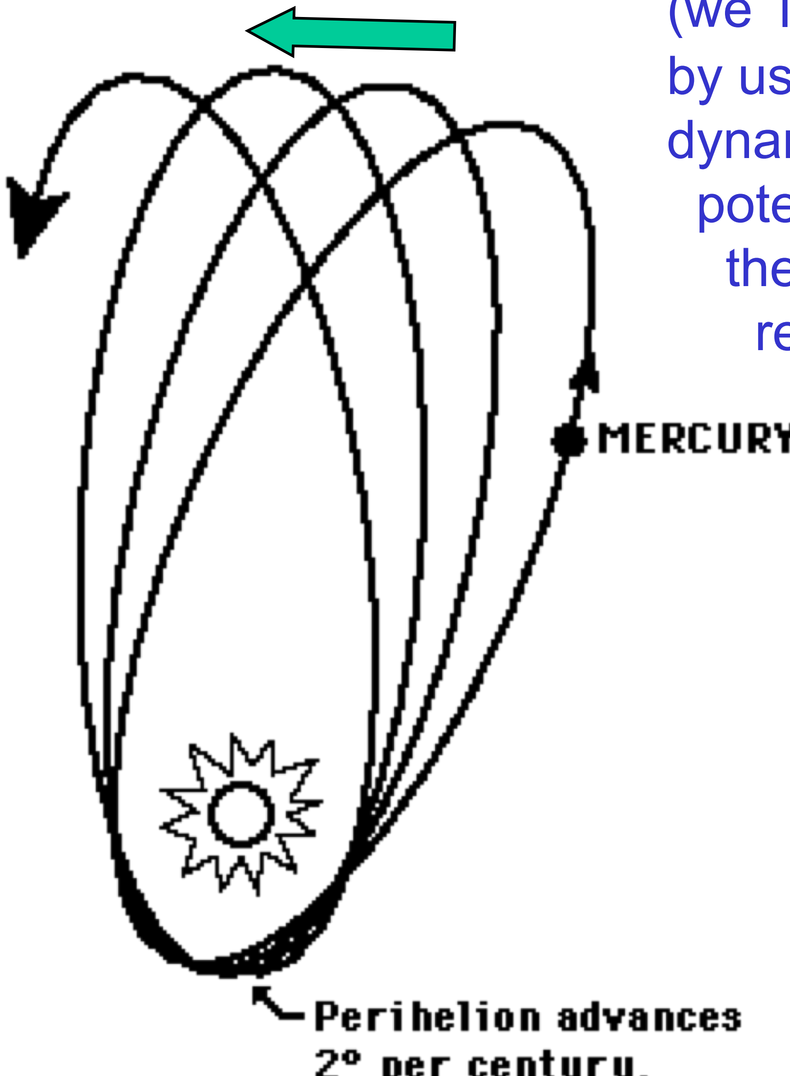
$r = a(1-e^2)/(1+ e\cos\theta)$

The <u>relativistic</u> precession of orbits

as one of the applications of general perturbation theory



(1879-1955)



(we'll cheat a little by using Newtonian dynamics with a modified potential, approximating the use of general relativity; that kind of cheating is quite OK!).

(drawing not to scale, shape and the precession rate exaggerated!)

General theory of perturbations & the relativistic precession of the orbits of Mercury, and Υ And b

$$f_r = -\frac{GM}{r^2} \underbrace{- \frac{3GM\ell^2}{c^2 r^4}}_{R \text{ (radial accel.)}}$$

$\dot{\omega}$ 

Gauss' theory in case of $T=0$

$$\dot{\omega} = -\frac{\sqrt{1-e^2}}{n a e} R \cos\theta \qquad \left(n = \sqrt{\frac{GM}{a^3}}, \; n = 2\pi/P\right)$$

or, using $r^2 \frac{d\theta}{dt} = \ell = \text{const}$,

time-averaged $\langle\dot{\omega}\rangle$ is

$$\langle\dot{\omega}\rangle = \frac{1}{P}\int_0^P \dot{\omega}\, dt = \frac{n}{2\pi}\int_0^{2\pi} \dot{\omega}\, \frac{r^2}{\ell}\, d\theta$$

time-averaged $\langle\dot{\omega}\rangle$ is

$$\langle\dot{\omega}\rangle = \frac{1}{P}\int_0^P \dot{\omega}\,dt = \frac{n}{2\pi}\int_0^{2\pi} \dot{\omega}\,\frac{r^2}{\ell}\,d\theta$$

$$= +\frac{3GM\ell\sqrt{1-e^2}}{2\pi a e c^2}\int_0^{2\pi} \cos\theta\,\frac{d\theta}{r^2}$$

We have $\int_0^{2\pi} \cos\theta\,(1+e\cos\theta)^2\,d\theta = 2\pi e$,

hence (denoting $GM/c^2 =: r_g$)

$$\boxed{\langle\dot{\omega}\rangle = \frac{3n}{1-e^2}\cdot\frac{r_g}{a}}$$

Sun's gravitational radius $r_g = \frac{GM_\odot}{c^2} = 1.47\,\mathrm{km}$

Keplerian velocity at $r = r_g$ would equal the speed of light, c.
(Black hole inside r_g !)

For Mercury $a = 5.79 \cdot 10^7$ km,
$e = 0.206$

$\langle \dot{\omega} \rangle = 42.''98$ / century

Even there, this is but a small part of the total precession equal $\sim 5600''$/century, mostly due to planets.

But this was the missing puzzle piece! ($41.''4 \pm 0.''9$ was missing.)

There is an extrasolar planetary system where relativistic precession saved a planet : Υ And = "upsilon Andromedae".

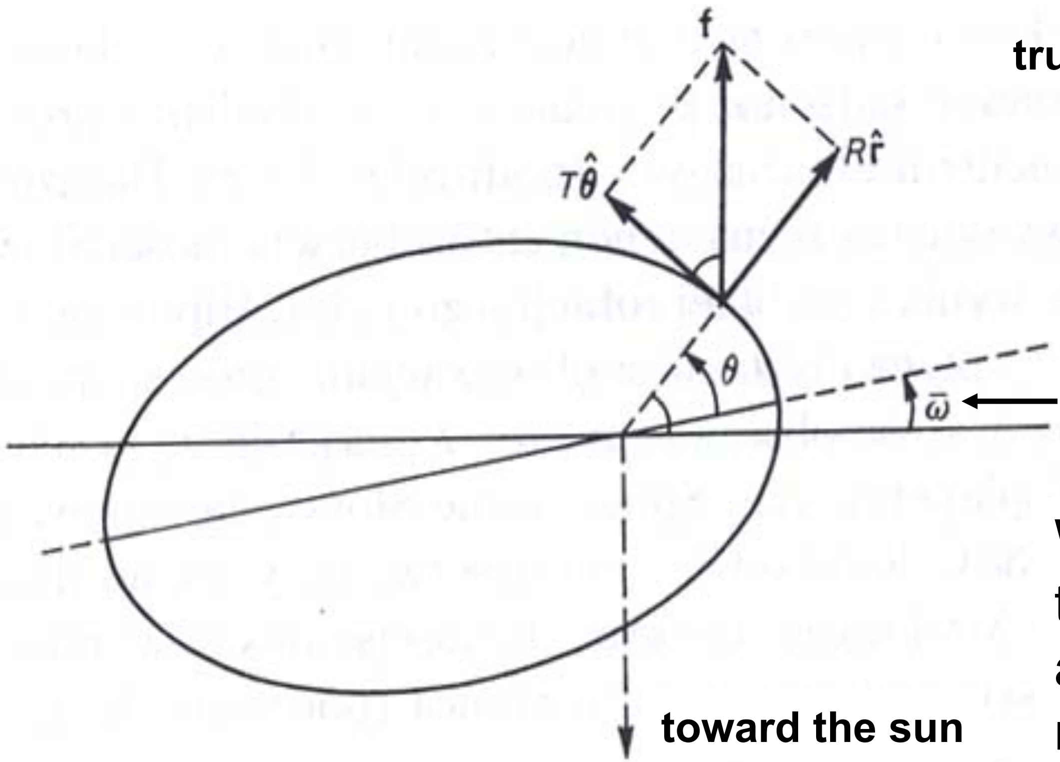
(3 Gyr old)

this inner, Jupiter-class planet would align its orbit with that of its two larger neighbors and then a growing eccentricity would cause its collision with Υ And. Relativistic precession prevents the alignement, so that $e \approx 0$.

General theory of pert. & the problem of a satellite illuminated by the star ("solar sailboat")

$R = f\sin(\theta+\varpi), \quad T = f\cos(\theta+\varpi)$

true anomaly (orbital angle)



We will use solar sail problem to illustrate three different approaches to celestial mechanics: two perturbation theories and the energy method

Gauss' theory averaged over orbital period gives the following formulae, found by the computer algebra program *Mathematica* (or Maple, ...)

$$\langle \dot{a} \rangle = \frac{2fa^2(1-e^2)^{3/2}}{l} \cdot \frac{1}{2\pi}\int_0^{2\pi} \frac{e+\cos\theta}{(1+e\cos\theta)^2}\, d\theta = 0,$$

$$\langle \dot{e} \rangle = +\frac{3f}{2na}\sqrt{1-e^2}\cos\varpi$$

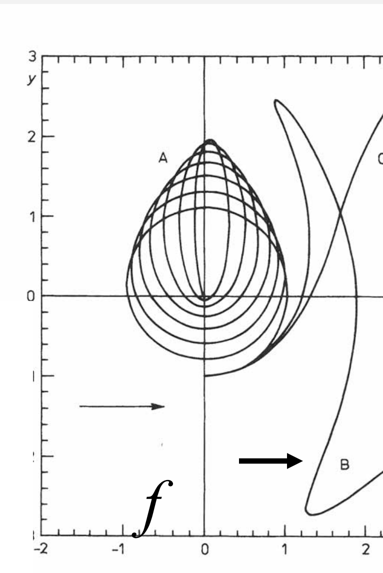
$$\langle \dot{\varpi} \rangle = -\frac{3f}{2nae}\sin\varpi$$

$\Rightarrow \varpi = 0$ is a stable equilibrium orientation of the orbit

In that configuration $\cos\varpi = 1$ and $\frac{\langle \dot{e} \rangle}{\sqrt{1-e^2}} = \frac{3f}{2na} > 0$, eccentricity increases steadily,

e(t) = *sin*(t/t_e), where t_e = (2na)/(3f)

Eventually, e ⟶1 after time (π/2)t_e. During this evolution, the orbit's orientation is *perpendicular* to the force from the sun!



Elongation of the orbit might destroy the satellite. Magnitude of $\dot{e} \sim f/v_K$, depends on the size of satellite via $f = f(s) \underset{\text{often}}{\sim} 1/s$. Dust most affected, planets very little.

Satellite can be saved by precession, and thus rapidly oscillating term $\dot{e} \sim \cos\tilde{\omega}(t)$, e.g. due to flattening of the planet caused by its rotation (spin):

$$\dot{\tilde{\omega}}_{prec} = + \frac{3 J_2}{2} \left(\frac{R}{a}\right)^2 n$$

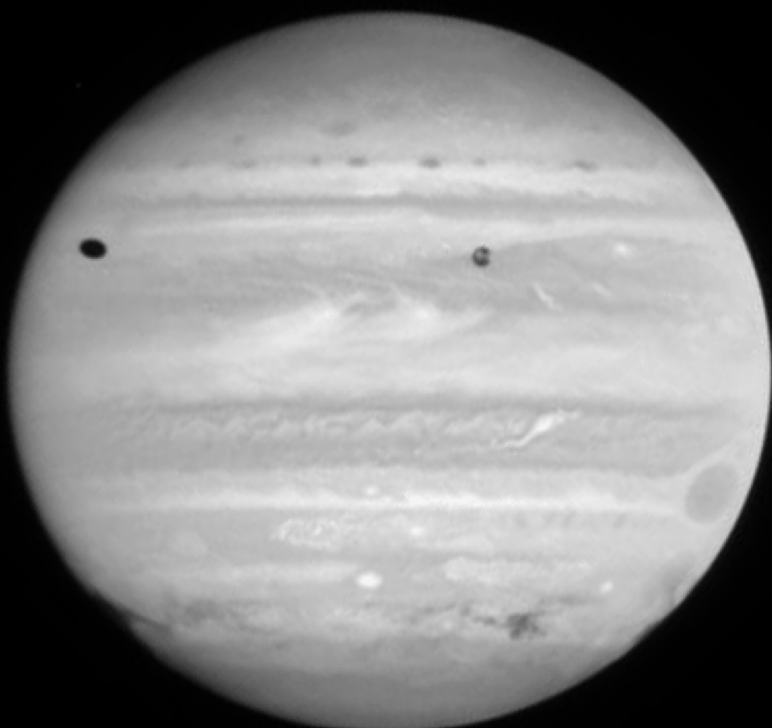
where $\dot{\tilde{\omega}}_{prec}$ = precession rate,

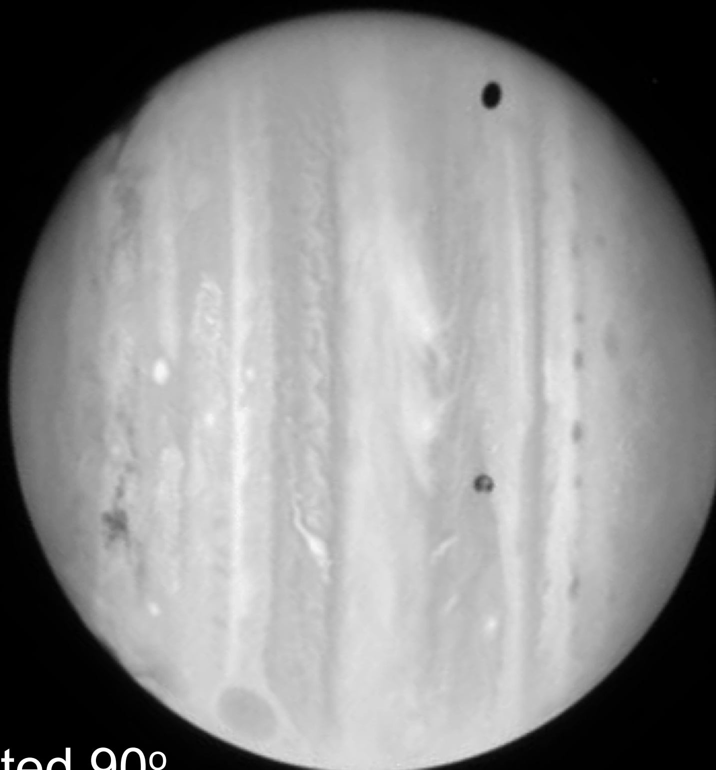
J_2 = non-dimensional coefficient describing the quadrupole correction to point-mass gravitational potential.

R = equatorial radius of a planet

$n = \sqrt{GM/a^3}$ ⟸ mean motion of satell. with semi-major axis a.

Jupiter’s rotational flattening is unmistakably seen here





the same image, rotated 90^o

Saturn has a slightly larger flattening.

Planetary rings only around giant planets!

One can show that indeed, dust in orbit around the Earth ($J_2 = 1.083e-3$), Venus ($J_2 = 6e-6$), Mars ($J_2 = 1.959e-3$), Moon ($J_2 = 2.024e-4$) is destroyed, whereas dust around Jupiter ($J_2 = 1.473e-2$), Saturn ($J_2 = 1.646e-2$) Uranus ($J_2 = 3.35e-3$) and Neptune ($J_2 = 4.1e-3$) is <u>saved</u> from that fate by rotational flattening of the planet and the above "spin-orbit" effect.

L8 Special theory of perturbations (numerical calculations)

Popular numerical integration methods for ODEs:
Euler method (1st order) & Runge-Kutta (2nd - 8th order)
Symplectic methods

Leonard Euler

Carle Runge

1856-1927

–

Martin Kutta

1867-1944

The Euler method

We want to approximate the solution of the differential equation

$$y'(t) = f(t, y(t)), \qquad y(t_0) = y_0, \qquad (1)$$

For instance, the Kepler problem which is a 2nd-order equation, can be turned into the 1st order equations by introducing double the number of equations and variables: e.g., instead of handling the second derivative of variable x, as in the Newton's equations of motion, one can integrate the first-order (=first derivative only) equations using variables x and v_x = dx/dt (that latter definition becomes an additional equation to be integrated).

Starting with the differential equation (1), we replace the derivative *y*' by the finite difference approximation, which yields the following formula

which yields

$$y'(t) \approx \frac{y(t+h) - y(t)}{h}, \qquad (2)$$

$$y(t+h) \approx y(t) + hf(t, y(t)) \qquad (3).$$

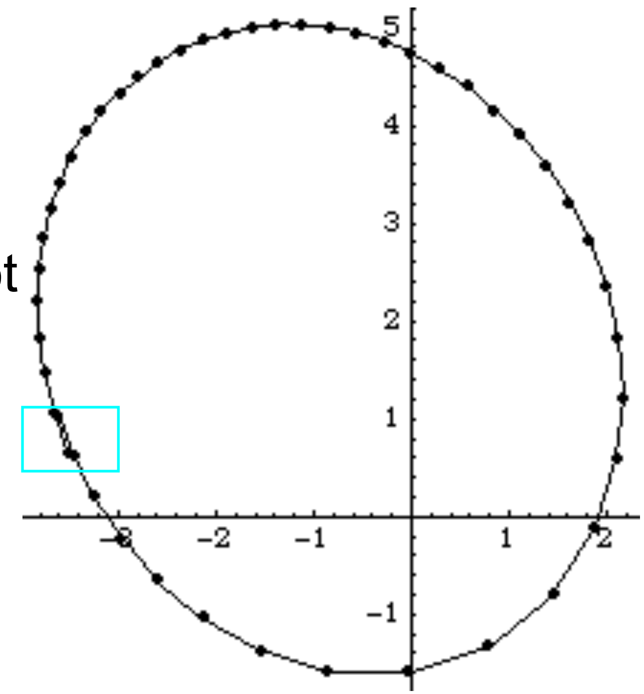
This formula is usually applied in the following way.

The Euler method (cont’d)

$$y(t+h) \approx y(t) + hf(t, y(t)) \qquad (3).$$

This formula is usually applied in the following way.
We choose a step size h, and we construct the sequence t_0, $t_1 = t_0 + h$, $t_2 = t_0 + 2h$, ... We denote by $\boldsymbol{y_n}$ a numerical estimate of the exact solution $y(t_n)$. Motivated by (3), we compute these estimates by the following recursive scheme

$$\boldsymbol{y_{n+1} = y_n + h\, f(t_n, y_n)}.$$



This is the *Euler method* (1768), discovered but not formally published 10^2 yr earlier by Robert Hook.
It’s a first order method, meaning that the total error is $\sim h^1$. It requires small time steps & has mediocre accuracy, but it’s very simple!

The classical fourth-order Runge-Kutta method

One member of the family of Runge-Kutta methods is so commonly used, that it is often referred to as "RK4" or simply as "*the* Runge-Kutta method". The RK4 method for the problem

$$y'(t) = f(t, y(t)), \qquad y(t_0) = y_0, \qquad (1)$$

is given by the following equation:

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where

$$k_1 = f(t_n, y_n)$$

$$k_2 = f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_1\right)$$

$$k_3 = f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_2\right)$$

$$k_4 = f(t_n + h, y_n + hk_3)$$

The interval h in orbital calculations is actually the timestep Δt.

Thus, the next value (y_{n+1}) is determined by the present value (y_n) plus the product of interval *h* and an estimate of space & time-averaged full time derivative of function y(t).

Runge-Kutta 4th order (continued)

Next value (y_{n+1}) is determined by the present value (y_n) an estimated average derivative or slope. That is a particular unevenly weighted average

$$\text{slope} = \frac{k_1 + 2k_2 + 2k_3 + k_4}{6}.$$

- k_1 is the slope at the beginning of the interval;
- k_2 is the slope at the midpoint of the interval, using k_1 to determine the value of y at the point $t_n + h/2$, using Euler's method
- k_3 is again the slope at the midpoint, using improved slope k_2
- k_4 is the slope at the end of the interval

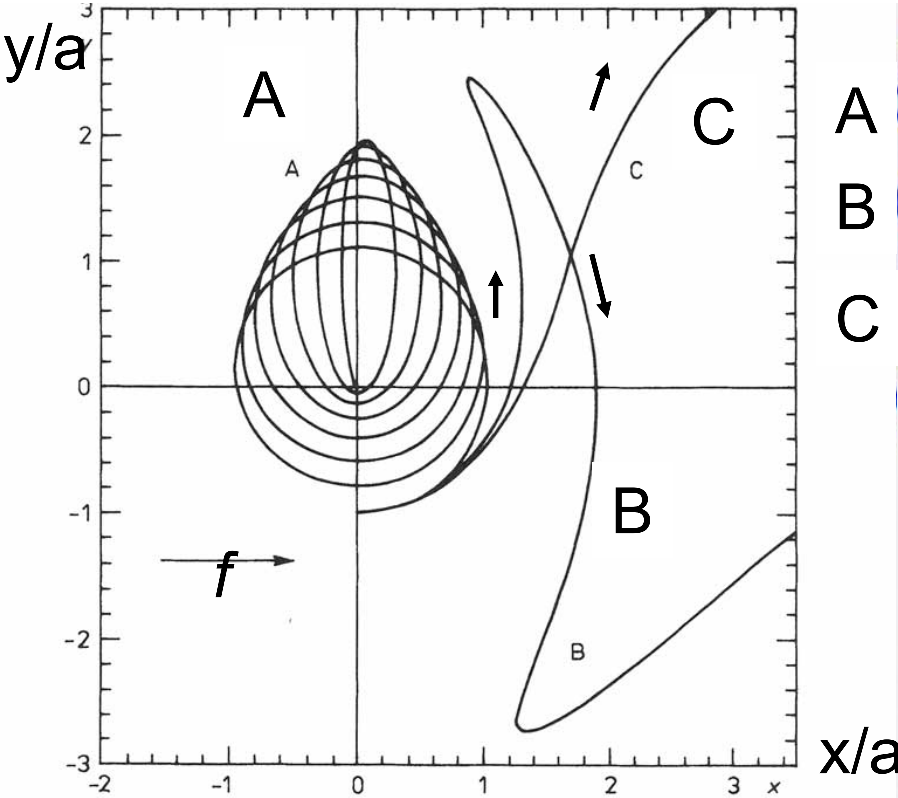
The RK4 method is a 4th order method, meaning that the 1-timestep error is $\sim h^5$, and global error over a finite time is $\sim h^4$. It allows larger time steps & better accuracy than 2nd order methods. But RK4 produces a gradually (slowly) increasing energy error, because it is not *symplectic*.

SYMPLECTIC METHODS

Leapfrog method is a 2nd order symplectic method. It looks like Euler method, but all the positions and velocities are separated in time by $\Delta t/2$ (so the integration needs to be carefully started and ended), and velocity component must be updated before position components.

Symplectic 4th order integrators exist (some require only 3, instead of 4 force evaluations per timestep!). They should be used in long-term integrations of Hamiltonian systems.

Solar sail problem revisited: case A



A $f = 0.02 \cdot f_0$

B $f = 0.134 \cdot f_0$

C $f = 0.2\, f_0$

$\left(f_0 = \frac{GM}{a^2}\right)$

Newton's equations of motion

$$\ddot{\vec{r}} = -\frac{GM}{r^2}\cdot\frac{\vec{r}}{r} + \vec{f}$$

or

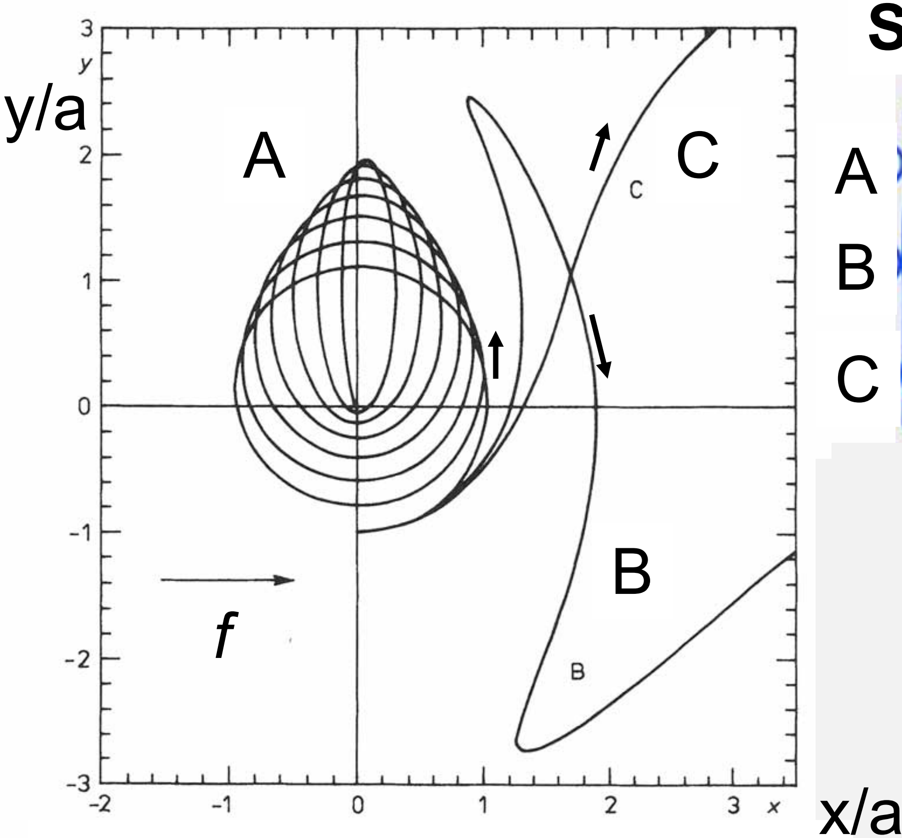
$$\begin{cases} \frac{d^2x}{dt^2} = -\frac{GMx}{r^3} + f \\ \frac{d^2y}{dt^2} = -\frac{GMy}{r^3} \end{cases}$$

Numerical integration
(Euler method, h = dt = 0.001 P)

Comparing the numerical results with analytical perturbation theory we see a good agreement in case A of small perturbations, f << 1. In this limit, analytical results are more elegant and general (valid for every f) than numerical integration:

Reminder: $e(t) = \sin t/t_e$, where $t_e = (2\, n\, a) / (3f)$, for arbitrary f, n, & a.

Solar sail problem revisited: B, C



Numerical integration
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A $f = 0.02 \cdot f_0$

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$$\left(f_0 = \frac{GM}{a^2}\right)$$

Newton's equations of motion

$$\ddot{\vec{r}} = -\frac{GM}{r^2} \cdot \frac{\vec{r}}{r} + \vec{f}$$

or

$$\begin{cases} \dfrac{d^2x}{dt^2} = -\dfrac{GMx}{r^3} + f \\ \dfrac{d^2y}{dt^2} = -\dfrac{GMy}{r^3} \end{cases}$$

However, in cases B and C of large perturbations, f ~ 0.1…1. In this limit, analytical treatment cannot be used, because the assumptions of the theory are not satisfied (changes of orbit are not gradual). Eccentricity becomes undefined after a fraction of the orbit (case B, C).

In this case, the computer is your best friend, though it requires a repeated calculation for each f, and always introduces numerical errors of 2 or 3 sorts: truncation (discretization) error, round-off error, and possible coding bugs.

INTEGRALS OF MOTION, STABILITY, CHAOS

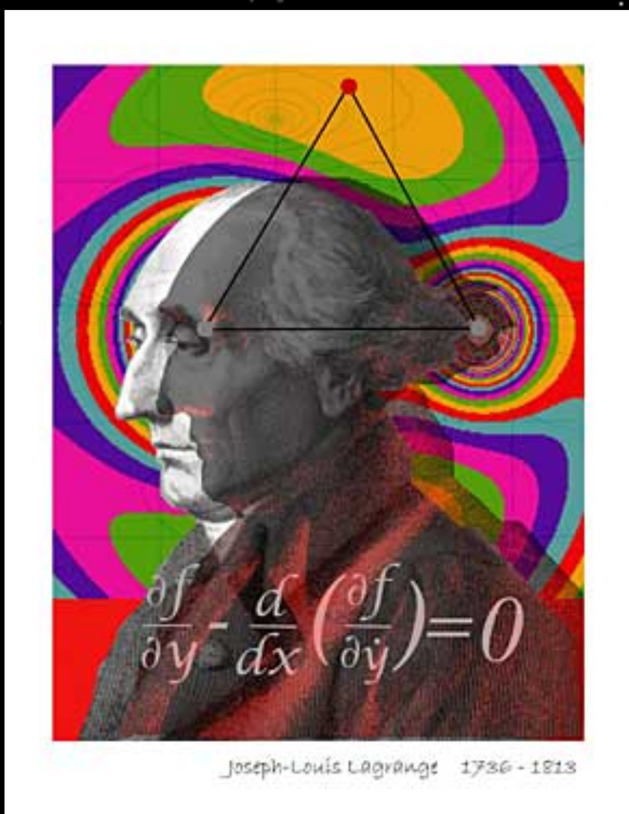
Lecture 8 ASTC25

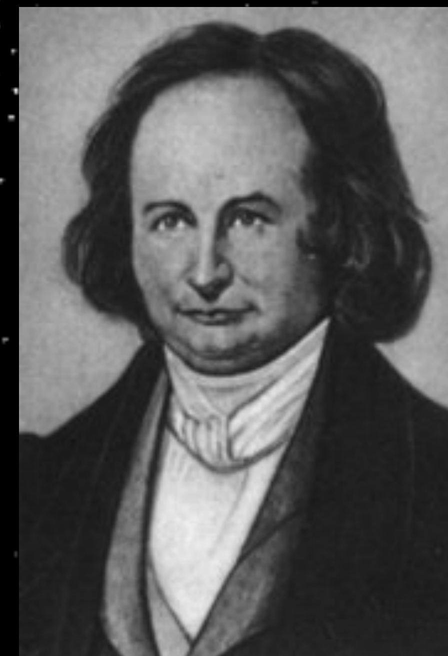
1. Energy methods (Integrals of motion)
2. Zero Velocity Surfaces (Curves)
3. R3B problem and the Roche lobe radius calculation
4. Lagrange points and their stability
5. Hill problem and Hill stability of orbits
6. Resonances
7. Chaos & stability in the Solar System
8. Corotation Region's width

Non-perturbative methods (energy constraints, integrals of motion)

in the

3 Body problem





Karl Gustav Jacob Jacobi (1804-1851)

Solar sail problem again

Integrals of motion; energetic method

$$\begin{cases} \ddot{x} = -\dfrac{GMx}{r^3} + f & \Big|\cdot \dot{x} \\ \ddot{y} = -\dfrac{GMy}{r^3} & \Big|\cdot \dot{y} \end{cases}$$

A standard trick to obtain energy integral

(+)

$$\dot{x}\ddot{x} + \dot{y}\ddot{y} = -\frac{x\dot{x}+y\dot{y}}{r^3}GM + f\dot{x}$$

$$\tfrac{1}{2}(\dot{x}^2)^{\cdot} + \tfrac{1}{2}(\dot{y}^2)^{\cdot} = -\underbrace{\vec{v}\cdot\nabla\Phi}_{\vec{v} = (\dot{x},\dot{y})} + (fx)^{\cdot}$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla,$$

$$-\vec{v}\cdot\nabla\Phi = -\dot{\Phi}$$

$$\left(\Phi = -\frac{GM}{r}\right)$$

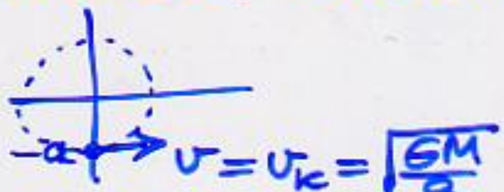
$$\boxed{\tfrac{1}{2}v^2 + \Phi - fx = C = \text{const}}$$

energy integral for equations of motion of a "solar sailboat", a.k.a. Jacobi constant.

$$\boxed{\frac{1}{2}v^2 + \Phi - fx = C = \text{const}}$$

energy integral for equations of motion of a "solar sailboat", a.k.a. **Jacobi constant.**

Consider our initial condition:



$$C = \frac{v_0^2}{2} - \frac{GM}{a} - f \cdot 0 = \left(\frac{1}{2} - 1\right)\frac{GM}{a} = -\frac{1}{2} \cdot v_0^2$$

The particle will keep that value forever.
If it gets close to body M, $r, x \ll a_0$ and

$$v^2 = \left(\frac{2GM}{r} + 2fx + 2C\right) \gg v_0^2$$

But if it departs from M, at some point(s), called ZERO VELOCITY CURVE (ZVC) or SURFACE, v^2 drops to zero and the trajectory has to turn back.

Energy criterion guarantees that a particle cannot cross the Zero Velocity Curve (or surface), and therefore is stable in the Jacobi sense (energetically).

However, remember that this is particular definition of stability which allows the particle to physically collide with the massive body or bodies -- only the escape from the allowed region is forbidden! In our case, substituting v=0 into Jacobi constant, we obtain:

ZVC: $$\frac{1}{(x^2+y^2)^{\frac{1}{2}}} + fx = \frac{1}{2} = -\frac{C}{v_0^2}$$

$$y=0 \Rightarrow \quad 2fx^2 - x + 2 = 0$$

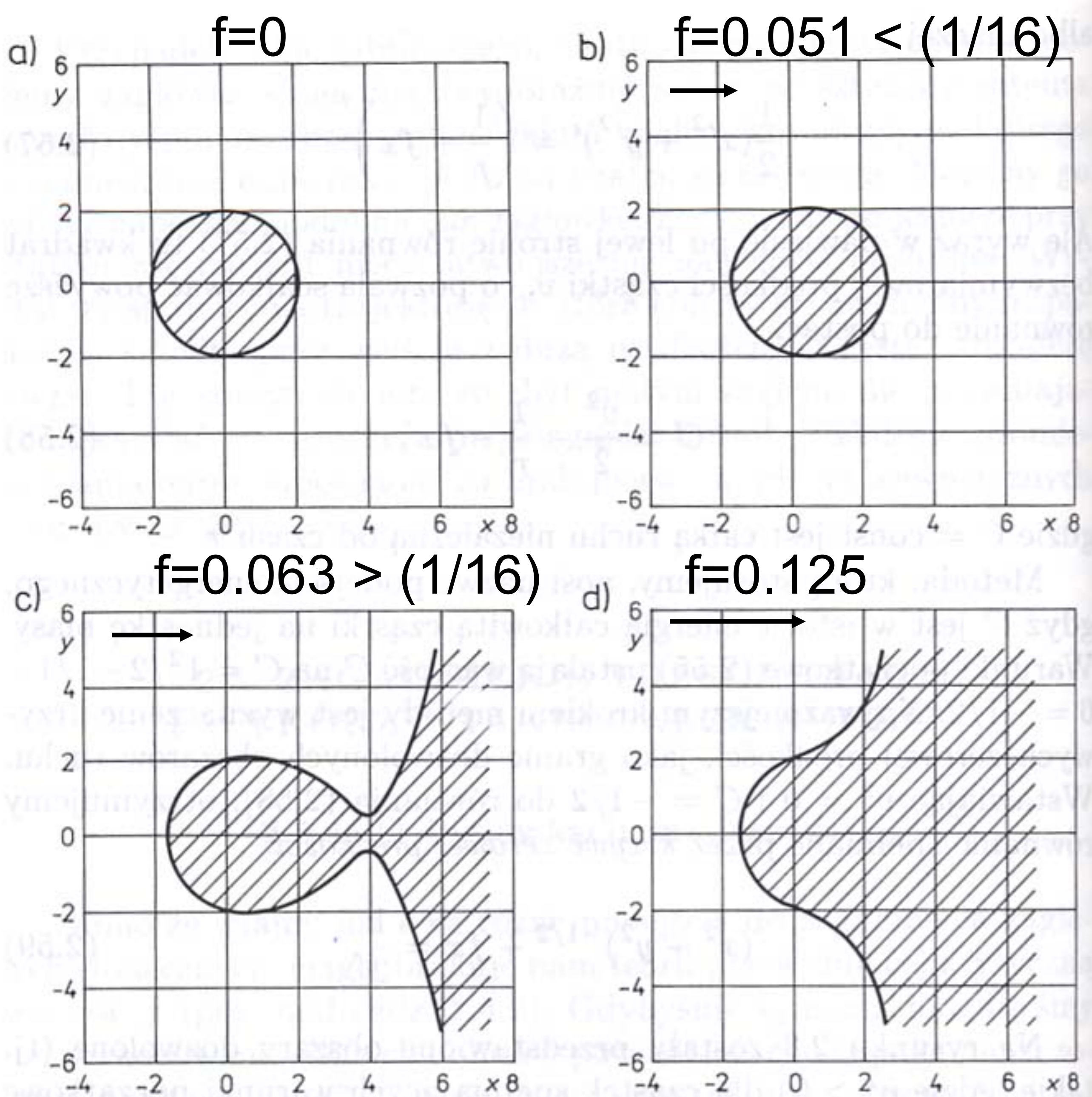
solutions $x>0$ only if

$$f < \frac{1}{16} = 0.0625$$

(no numerical simulations necessary!)

here, f is given in units of GM/r_0^2

Allowed regions of motion in solar wind (hatched) lie within the Zero Velocity Curve



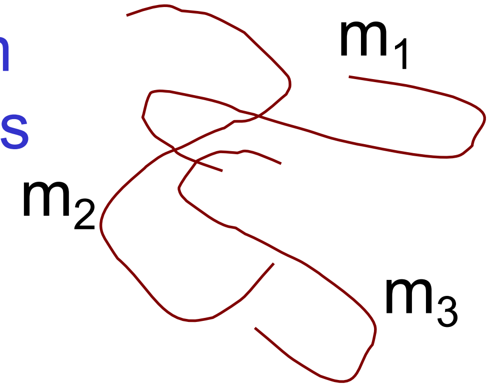
particle *cannot* escape from the planet located at (0,0)

particle *can* (but does not always) escape from the planet

(cf. numerical cases B and C, where f=0.134, and 0.2, much above the limit of f=1/16).

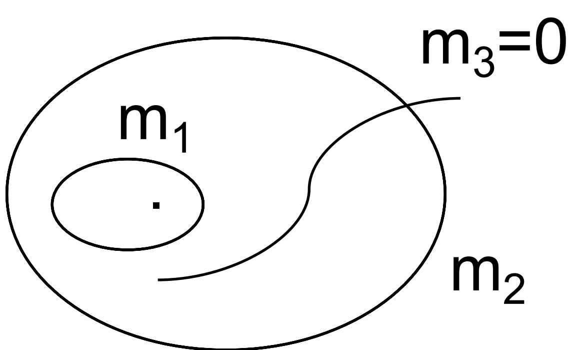
3 Body Problem (any masses & orbits)

3-body problems have no known closed-form (analytical) solution valid for all init. conditions

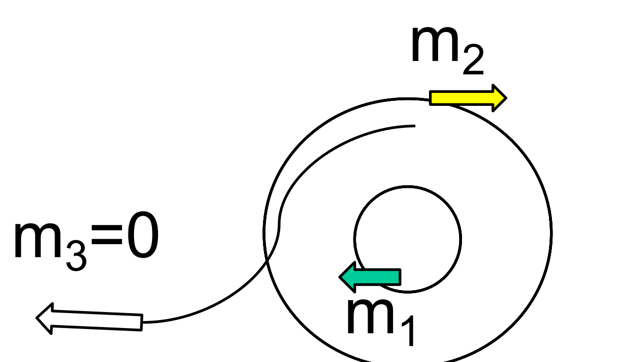


Restricted 3 Body Problem (3rd body massless, $m_3 = 0$)

Eccentric
(2 massive bodies $m_{1,2}$ on elliptic orbits)

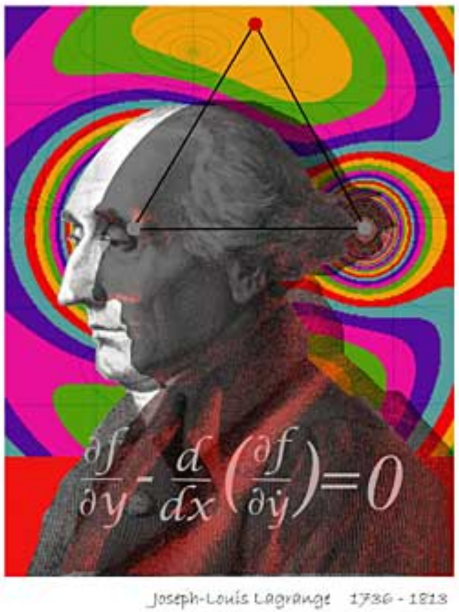


Circular
(2 massive bodies $m_{1,2}$ on circular orbits)



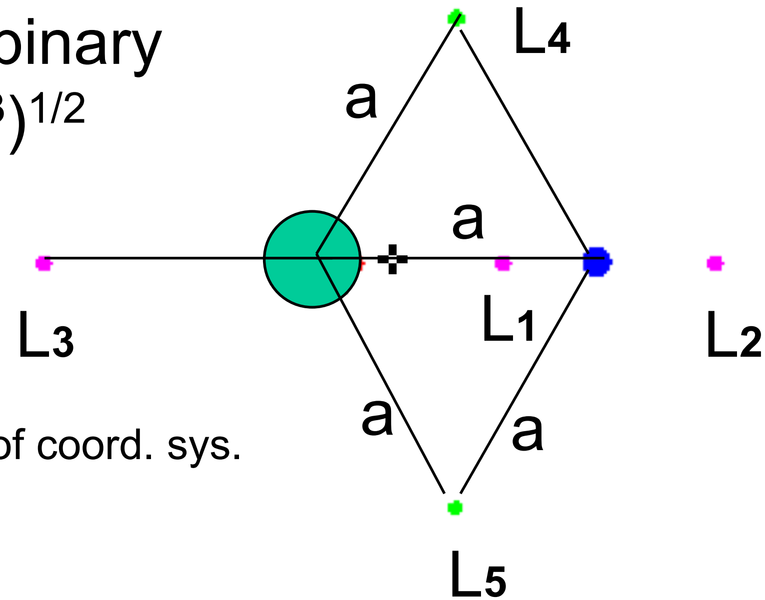
Circular Restricted 3-Body Problem (CR3B)

We sometimes talk about CR3B problem but call it R3B for short.



The frame rotation speed is the mean motion of the massive binary

$\Omega = n = (GM/a^3)^{1/2}$



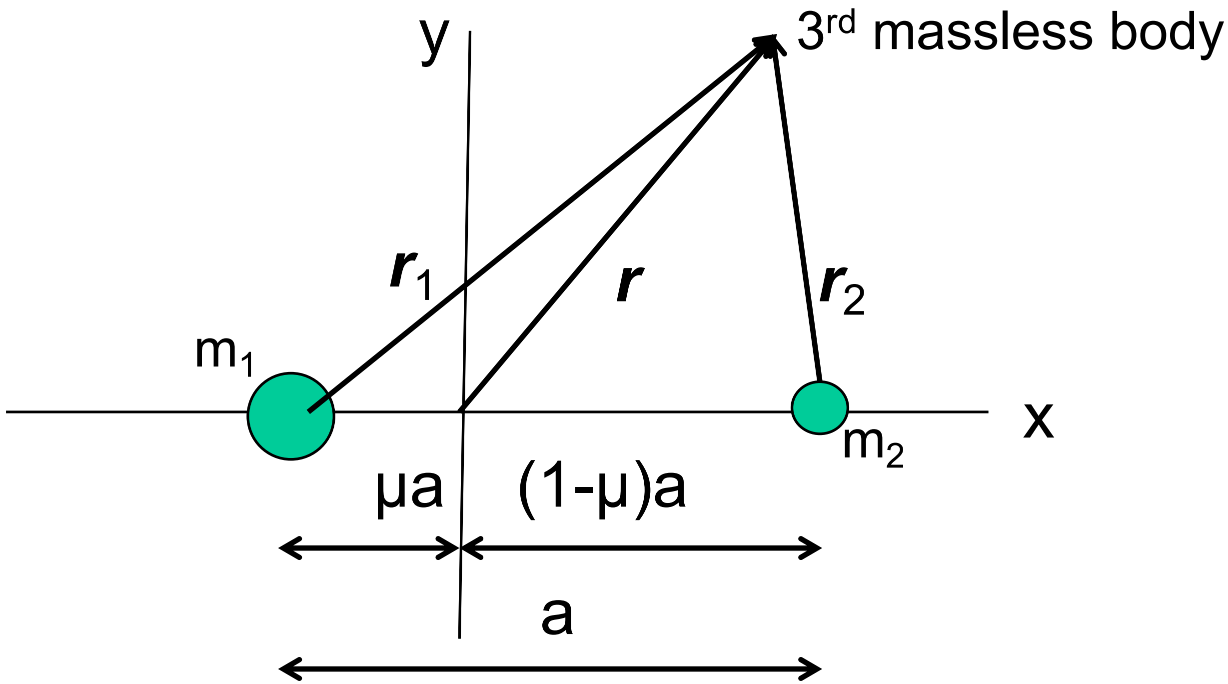
✣ = C.M. = origin of coord. sys.

Joseph-Louis Lagrange (1736-1813)
[Giuseppe Lodovico Lagrangia]

“Restricted” because the gravity of particle moving around the two massive bodies is neglected (so it’s a 2-Body problem plus 1 massless particle, whose 5 equilibrium positions are shown in the figure by small colored dots.) Furthermore, a circular motion of two massive bodies is assumed, so they stand still in the rotating frame. This gives us some important advantages.

Restricted 3 Body, Circular problem

Center of rotating coordinates (x,y,z) is chosen as the center of masses 1 and 2; mass parameter is $\mu = m_2/M = m_2/(m_1+m_2)$. Third body has position vector $\boldsymbol{r} = (x,y,z)$



The equation of motion of CR3B with $\boldsymbol{r} = (x,y,z)$ being position of the 3rd body in the frame rotating with two massive bodies, and velocity vector $\boldsymbol{v} = \dot{\boldsymbol{r}} = d\boldsymbol{r}/dt$, is

$$\ddot{\boldsymbol{r}} = -\nabla\Phi + \Omega^2 \boldsymbol{r} + 2\,\boldsymbol{\Omega} \times \boldsymbol{v} \quad or$$

$$d\boldsymbol{v}/dt = -\nabla\Phi + \Omega^2 \boldsymbol{r} + 2\,\boldsymbol{\Omega} \times \boldsymbol{v}$$

acceler. = gravity + centrifugal + Coriolis acc.

$\boldsymbol{f} = -\nabla\Phi$ stands for the gravitational force field (per unit mass of the test particle) due to bodies 1 & 2, derived from time-independent, scalar, gravitational potential $\Phi(\boldsymbol{r})$.

You can expand the above into 3 components

$$\ddot{x} = -d\Phi/dx + \Omega^2 x + 2\Omega\,\dot{y}$$

$$\ddot{y} = -d\Phi/dy + \Omega^2 y - 2\Omega\,\dot{x}$$

$$\ddot{z} = -d\Phi/dz$$

This is used e.g. in numerical solutions of the R3B equations. Notice that $\boldsymbol{r}(t)=(x,y,z)$ is all we seek in this problem. Positions of the massive bodies are known and unchanging, and Ω is the known angular speed of the binary, not of the 3rd body (test particle).

The derivation of energy (Jacobi) integral in CR3B does not differ much from the analogous derivation of energy conservation law in non-rotating systems: we also form the dot product of the equations of motion with velocity (***v***•...) and convert the l.h.s. (***v***•d***v***/dt) to full time derivative of specific kinetic energy d(½ ***v***•***v***)/dt. But on the r.h.s. we now have two additional accelerations (Coriolis and centrifugal terms) due to non-inertial, accelerated frame. Luckily the dot product of ***v*** and the Coriolis term, itself perpendicular to ***v***, vanishes: $\boldsymbol{v} \bullet (2\boldsymbol{\Omega} \times \mathbf{v}) = 0$.

The centrifugal term can be written as a gradient of a 'centrifugal scalar potential' $-\frac{1}{2}\,\Omega^2\, r^2$, since $-\nabla(-\frac{1}{2}\,\Omega^2 r^2) = \Omega^2\,\nabla(\frac{1}{2}\,\boldsymbol{r}\bullet\boldsymbol{r}) = +\Omega^2\boldsymbol{r}$, which added to the sum $\Phi = -Gm_1/r_1 - Gm_2/r_2$ of the grav. potentials of two bodies forms an effective (grav.+centr.) potential

$$\Phi_{eff} = -Gm_1/r_1 - Gm_2/r_2 - \tfrac{1}{2}\,\Omega^2\, r^2 .$$

For historical reasons, the effective potential of the R3B is often defined as a positive quantity $-2\Phi_{eff}$. If someone is using "Jacobi constant" look closely at the definition and if you see positive signs in $+Gm_i/r_i$ terms such as in the constant C below, then you know it's the historic definition.

MORE DETAILS, if you want them: Direct proof that effective potential is $\Phi_{eff} = \Phi - \frac{1}{2}\Omega^2 r^2 = -GM(1-\mu)/r_1 - GM\mu/r_2 - \frac{1}{2}\Omega^2 r^2$

where $r^2(t)=x^2+y^2+z^2$, $r_i^2(t)=(x-x_i)^2+y^2+z^2$ *,i.e.* $r_i(t)=[(x-x_i)^2+y^2+z^2]^{1/2}$ *and* x_i*=const. is the x coordinate of body number i = 1,2.*

Let's find $(-\Phi_{eff})^{\cdot}$. First, calculus gives full time derivative of $1/r_i$

$(1/r_i)^{\cdot} = -(1/r_i^2)\, dr_i/dt$. Each r_i changes because 3rd body moves and x,y,z depend on time; dr_i/dt is a sum of 3 changes:

$dr_i/dt = dr_i/dx\ dx/dt + dr_i/dy\ dy/dt + dr_i/dz\ dz/dt =$

$= -(x/r_i)\, v_x - (y/r_i)\, v_y - (z/r_i)\, v_z = (-\mathbf{r}/r_i)\bullet\mathbf{v},$ so we get

$(1/r_i)^{\cdot} = (\mathbf{r}/r_i^3)\bullet\mathbf{v} = \mathbf{v}\bullet\nabla(-1/r_i),$ because $\nabla(-1/r_i) = \mathbf{r}/r_i^3$.

We have two such terms with different constants in $(-\Phi_{eff})^{\cdot} =$ $-d\Phi_{eff}/dt$, plus one that looks like

$d[\frac{1}{2}\Omega^2 r^2]/dt = \frac{1}{2} d[\Omega^2 \mathbf{r}\bullet\mathbf{r}]/dt = = \Omega^2\, \mathbf{r}\bullet\mathbf{v},$ so finally

$(-\Phi_{eff})^{\cdot} = -\mathbf{v}\bullet\nabla\Phi + \mathbf{v}\bullet\Omega^2\mathbf{r} = \mathbf{v}\bullet[-\nabla\Phi eff\ ]$

Let's copy that result to the next page and compare with the equation of motion in R3B multipied (•) by ***v***

$(-\Phi_{eff})^{\cdot} = \boldsymbol{v} \bullet [-\nabla\Phi\text{eff}]$

while the eq. of motion reads

$$\boldsymbol{v}^{\cdot} = -\nabla\Phi + \Omega^2 \boldsymbol{r} + 2\boldsymbol{\Omega} \times \boldsymbol{v} = -\nabla\Phi_{\text{eff}} + 2\boldsymbol{\Omega} \times \boldsymbol{v} .$$

Doing $\bullet\boldsymbol{v}$ *(on both sides)* gives

$[\frac{1}{2}\, \boldsymbol{v} \bullet \boldsymbol{v}]^{\cdot} = -\boldsymbol{v} \bullet \nabla\Phi_{\text{eff}}$, the same as in the uppermost equation.

We conclude that $(\Phi_{eff} + \frac{1}{2}\, \boldsymbol{v} \bullet \boldsymbol{v})^{\cdot} = 0$.

This proves that Jacobi integral or Jacobi energy, defined as

$E_J = \Phi_{eff} + \frac{1}{2} v^2$ *is a constant, i.e. it's independent of time.*

E_J has the physically intuitive interpretation (potential plus kinetic energy per unit mass of test particle) and negative signs of gravitational energy terms.

But, as mentioned, honoring the historical choice made long ago, we define another form of the integral of motion,

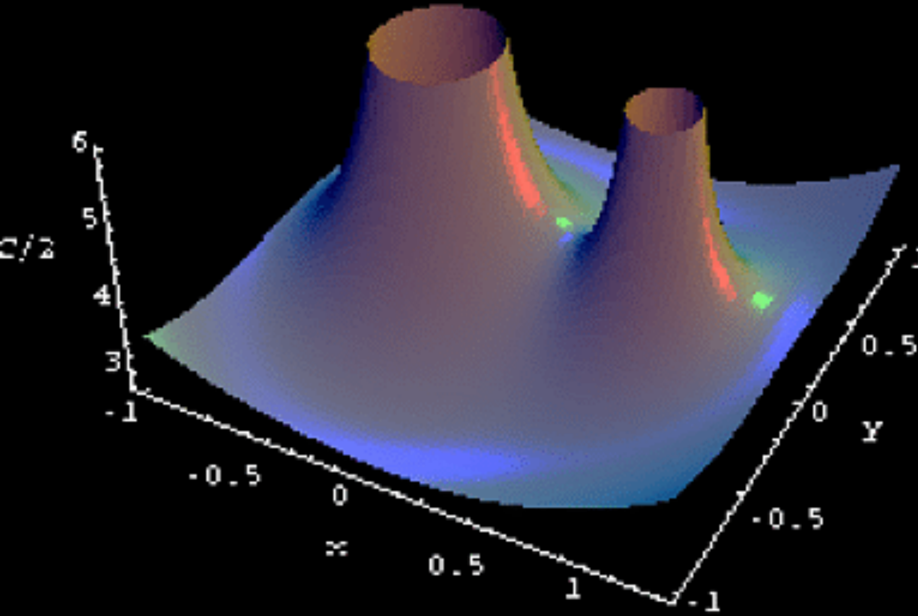
Jacobi constant C, as $C = -2E_J = -2\Phi_{eff} - v^2 = const.$

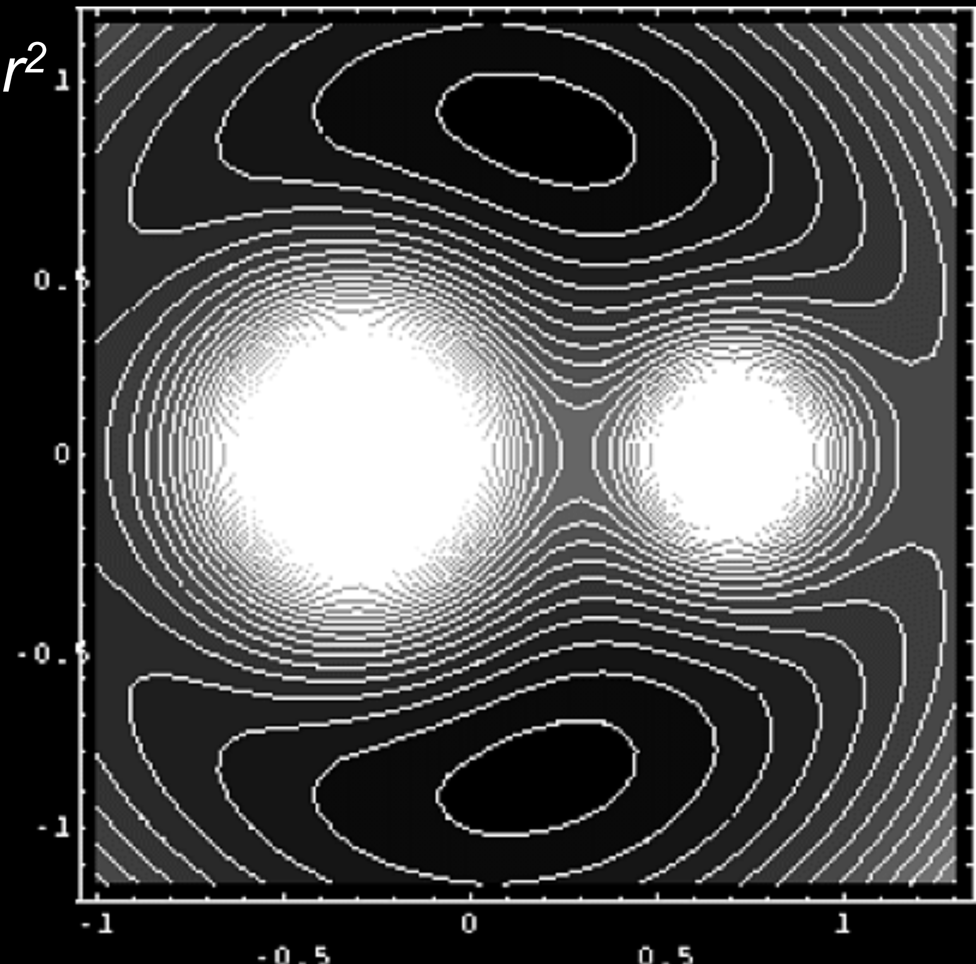
The values of Jacobi constants depend on initial position and speed of the 3rd body, but are conserved afterwards.

Effective potential in R3B

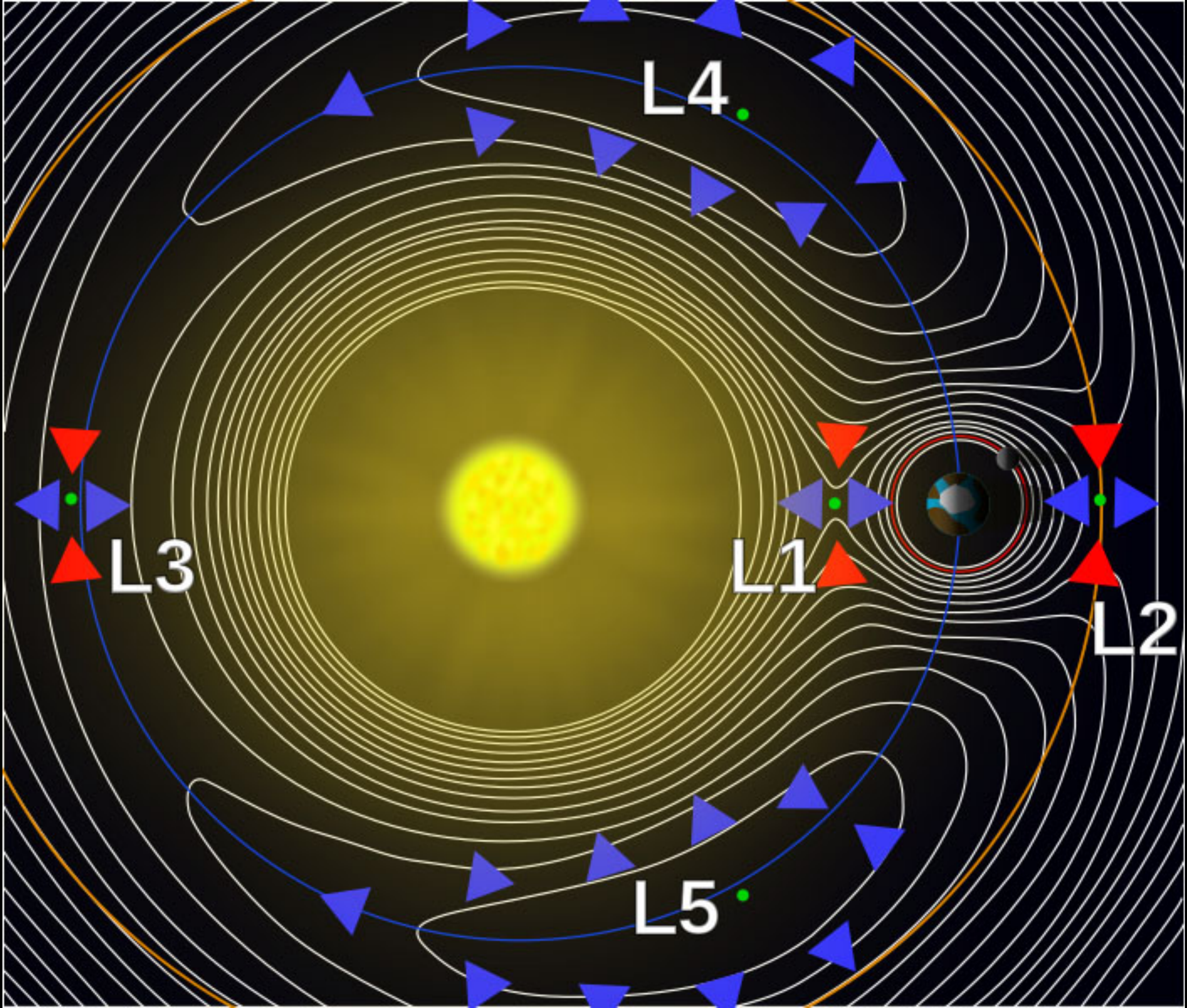
mass ratio = 0.2

$$-\Phi_{eff} = +Gm_1/r + Gm_2/r + \tfrac{1}{2}\,\Omega^2 r^2$$

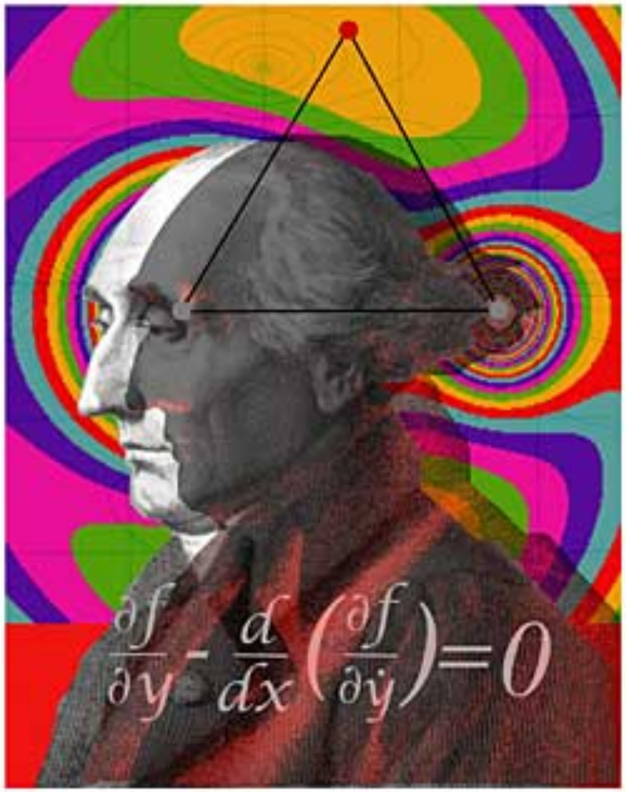




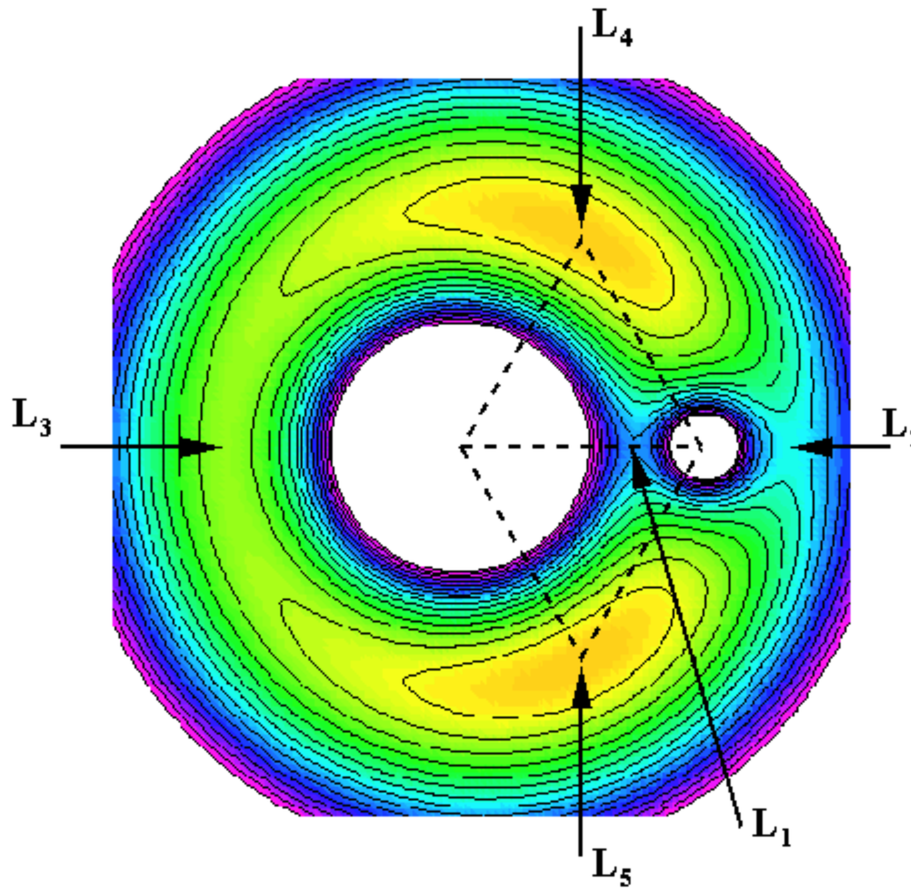
The historical effective potential of R3B is defined as negative of the Jacobi energy. Two gravitational potential wells around the two massive bodies thus appear as chimneys, and the centrifugal potential hill as a bowl outside.


L4
L3
L1
L2
L5

Lagrange points $L_1 \ldots L_5$ are equilibrium points in the circular R3B problem, which is formulated in the frame corotating with the binary system. Acceleration & velocity both equal zero there.



Joseph-Louis Lagrange 1736 - 1813



They are found at zero-gradient points of the effective potential of CR3B. Two of them are triangular points $L_{4..5}$ (extrema of potential). The 3 co-linear Lagrange points $L_{1..3}$ are saddle points of potential.

Jacobi integral and the topology of Zero Velocity Curves in R3B

$$\mu = m_1/(m_1+m_2)$$

$$\dot{\vec{r}} \cdot \ddot{\vec{r}} = n^2\, \dot{\vec{r}} \cdot \vec{r} + \dot{\vec{r}} \cdot \vec{f}_{grav}$$

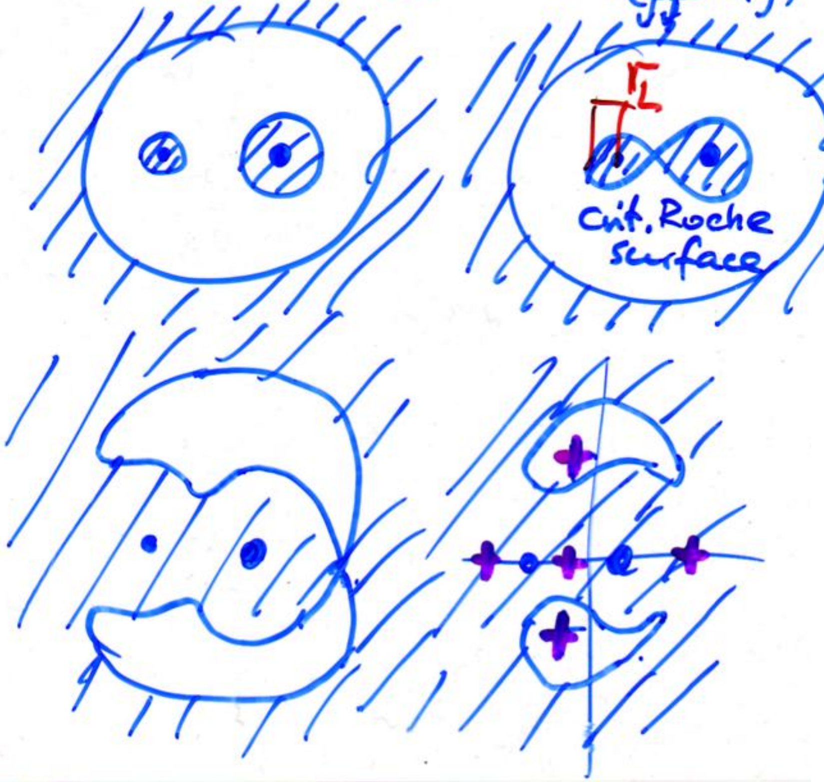
$$\tfrac{1}{2}(v^2)^{\bullet} = \tfrac{1}{2} n^2 (r^2)^{\bullet} - \dot{\Phi} \quad \Longrightarrow$$

where $\Phi = \left(\frac{1-\mu}{r_1} + \frac{\mu}{r_2}\right) v_0^2$ $\quad \left(v_0^2 = \frac{GM}{a}\right)$

$$C = -v^2 + 2\left(\Phi + \frac{n^2 r^2}{2}\right) = 2\Phi_{eff} - v^2$$

Jacobi integral

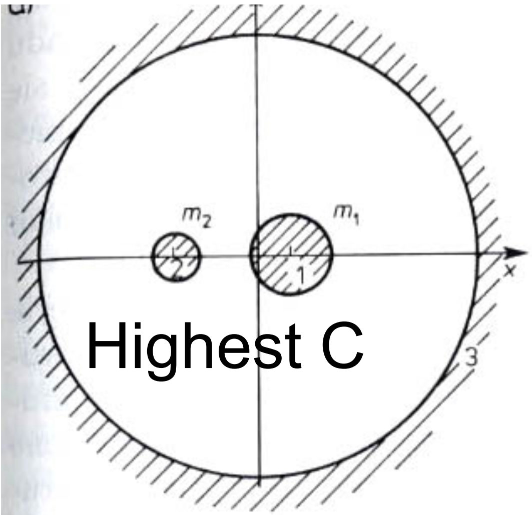
ZVC : $C = 2\,\Phi_{eff}(x,y)$

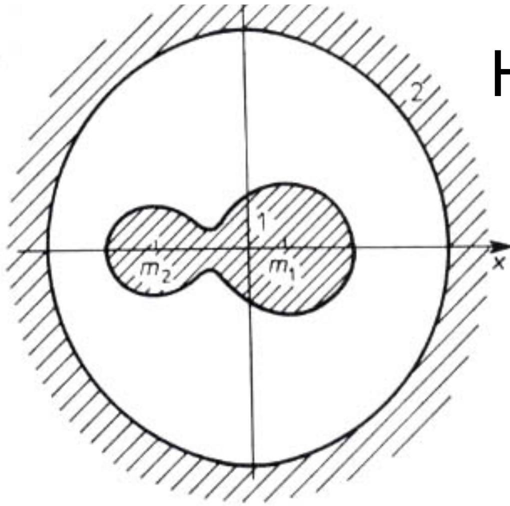


r_L = Roche lobe radius
(not uniquely defined, since there are 4 such radii)

+ : Lagrange points

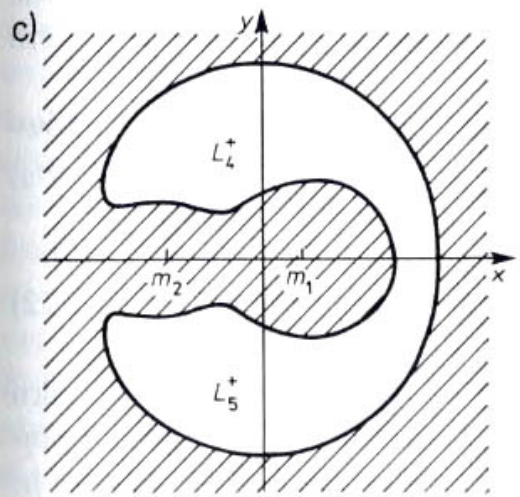
Sequence of allowed regions of motion (hatched) for particles starting with different C values (essentially, Jacobi constant ~ energy in corotating frame)

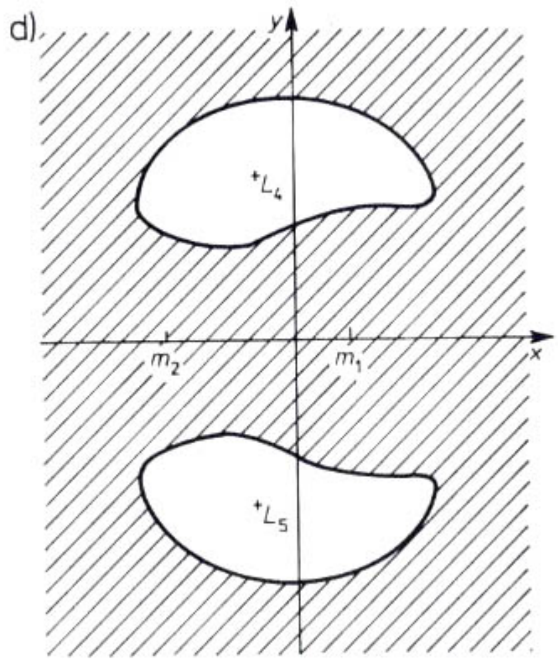




High C (e.g., particle starts close to one of the massive bodies)

Medium C





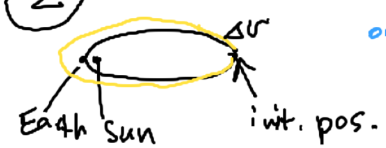
Low C (for instance, due to high init. velocity)

Notice a curious fact: regions near L_4 & L_5 are forbidden. These are potential maxima (taking a physical, negative gravity potential sign)

Tutorial 4:

1. Compute the distance x_L to “Lagrange” point in the solar sail problem
2. Compute the Jacobi constant at the saddle point of potential, at distance x_L
3. Prove that $f = (1/16)\,(GM/r_0)^{1/2}$ is the critical value allowing a passage through L pt.
4. Find the parameters (a,e) of the unperturbed and perturbed comet Dibiasky from movie “Don’t look up”. Assume initial perihelion distance 100 AU and aphelion distance 100000 AU. Assume the perturbation happens at the aphelium point and consistes of reduction of speed from v_{a0} to v_{a1}

(2) Comet Dibiasky



original orbit:

$r_{p0} = 100$ AU

$r_{a0} = 100\,000$ AU

aft. perturb.:

$r_{p1} = 1$ AU (!)

$r_{a1} = r_{a0}$

$v_{a0} = ?$

$v_{a1} = ?$

$\Delta v = ?$

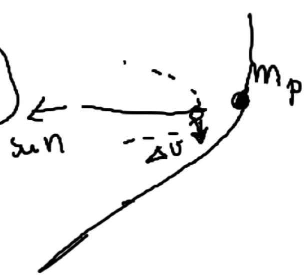
Q: Δv at r_{a0} that changes yellow (orig.) orbit into black orb.

$$\Rightarrow v_a = \sqrt{\frac{1-e}{1+e}}\sqrt{\frac{GM}{a}}$$

$$\left.\begin{aligned} 100\text{ AU} &= r_{p0} = a_0(1-e_0) \\ 100{,}000\text{ AU} &= r_{a0} = a_0(1+e_0) \end{aligned}\right\} \quad a_0 = \frac{100+10^5}{2}\text{ AU}, \quad e_0 = \frac{(10^5-10^2)\text{AU}}{2a_0} = \frac{10^5-10^2}{10^5+10^2} \approx 1-2\cdot10^{-3}$$

$$\left.\begin{aligned} r_{p1} &= a_1(1-e_1) \\ r_{a1} &= a_1(1+e_1) \end{aligned}\right\} \Rightarrow \begin{cases} a_1 = 50\,050\text{ AU} \\ e_1 = \dfrac{10^5-1}{10^5+1} \simeq 1-2\cdot10^{-5} \end{cases}$$

(3)



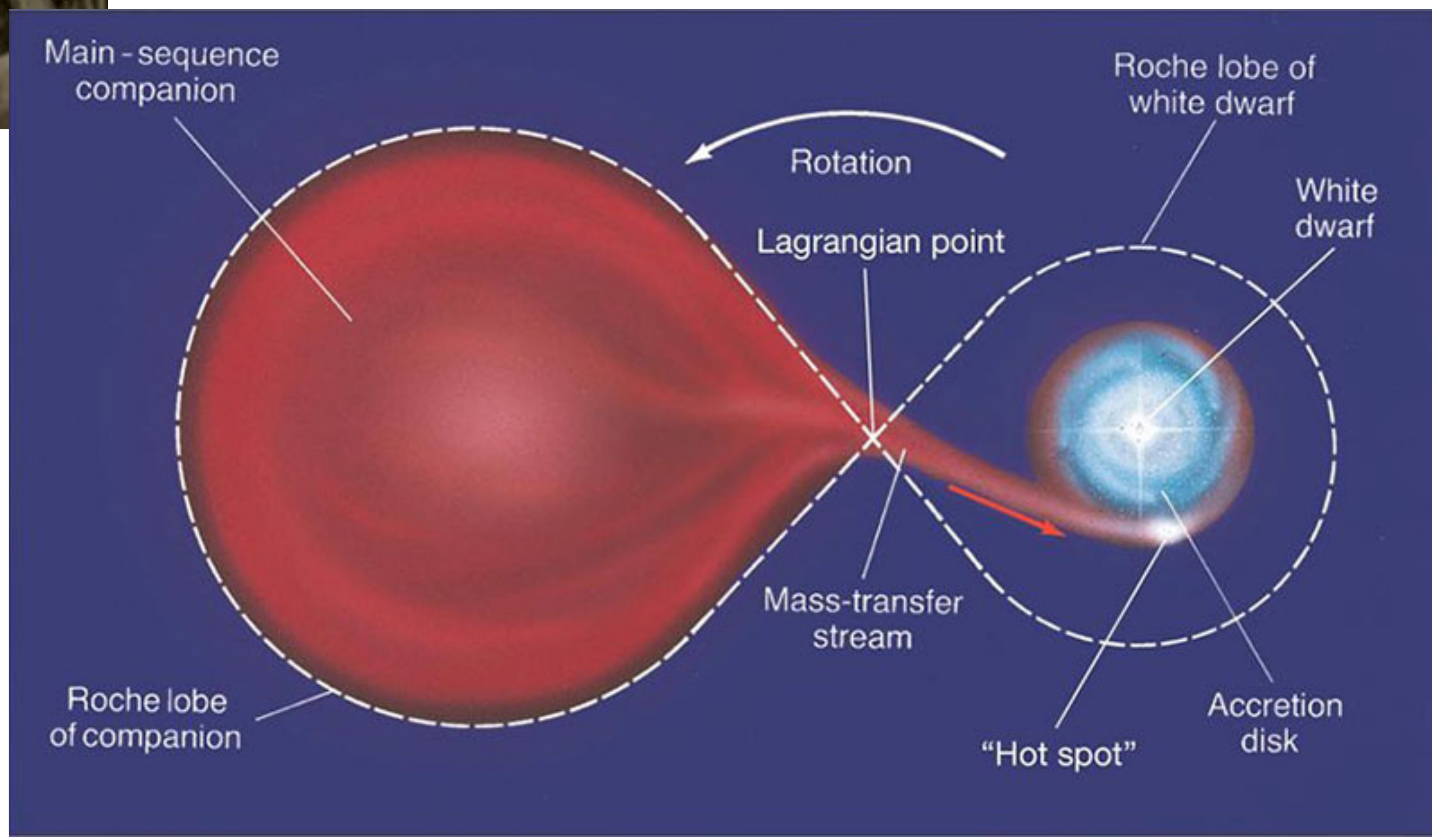
Note that $(1+e)a$ = same for both orbits $\equiv r_{a0}$

$$v_a = \sqrt{\frac{(1-e)GM}{r_{a0}}} \Rightarrow \frac{\Delta v}{v_{a0}} = \frac{\sqrt{1-e_0}-\sqrt{1-e_1}}{\sqrt{1-e_0}} \approx \frac{\sqrt{2\cdot10^{-3}}-\sqrt{2\cdot10^{-5}}}{\sqrt{2\cdot10^{-3}}} \approx$$

$\approx$ ...

THE CONCEPT OF ROCHE LOBE

Eduard A. Roche (1820-1883)
lived and taught at the University
in Montpellier, France



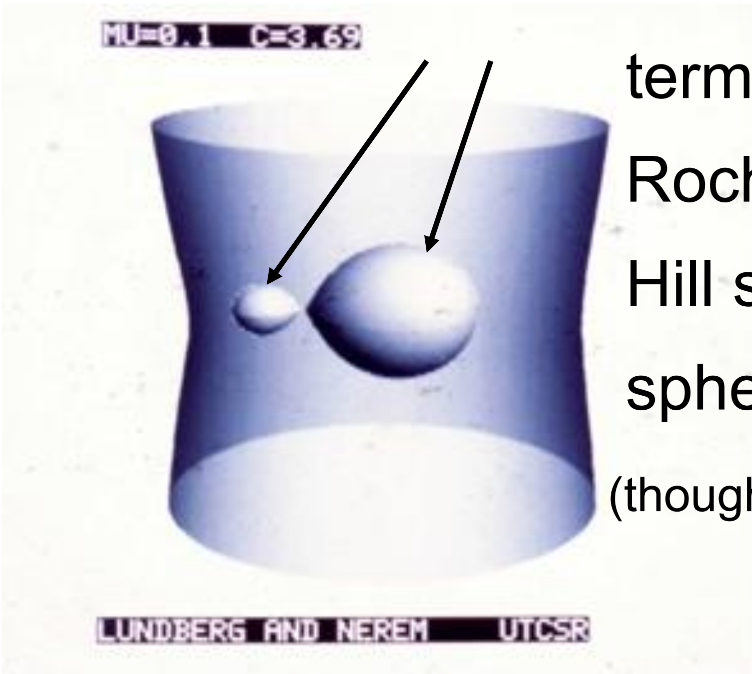
Mass ratio $\mu = m_2/(m_1+m_2) = 0.1$

C = R3B Jacobi constant with v=0

C=

3.60, 3.69. 4.00

Roche lobes



terminology:

Roche lobe ~

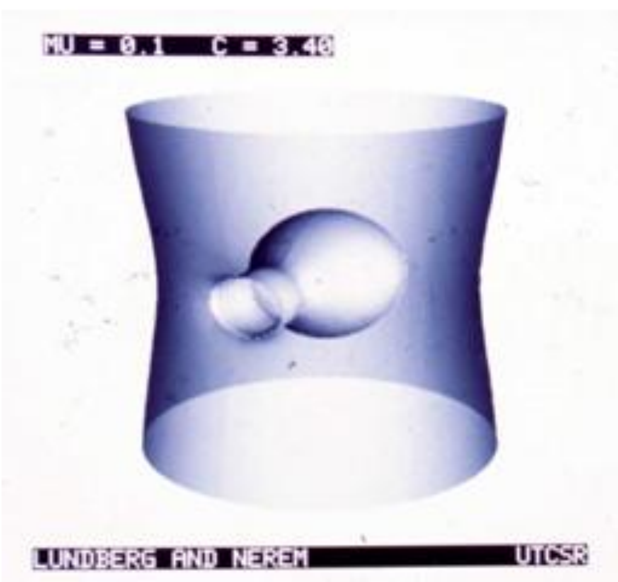
Hill sphere ~

sphere of influence

(though not really a sphere!)

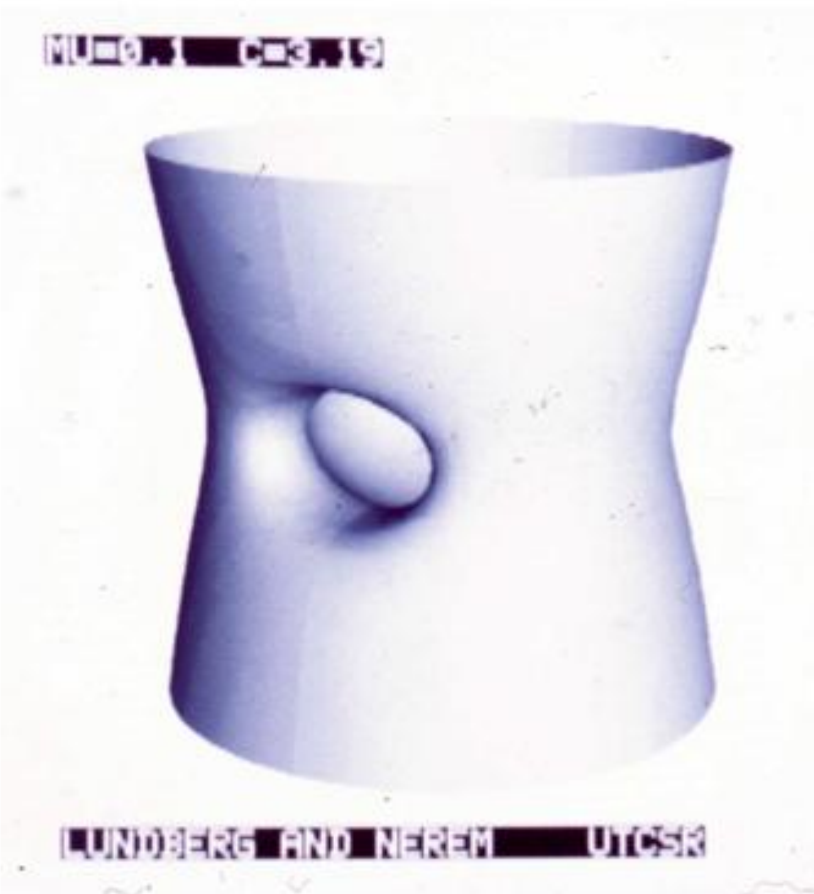
R3B problem. Mass ratio $\mu = m_2/(m_1+m_2) = 0.1$

C= 3.40



R3B problem. Mass ratio $\mu = m_2/(m_1+m_2) = 0.1$

C=3.19



Stability of (motion around) the L-points

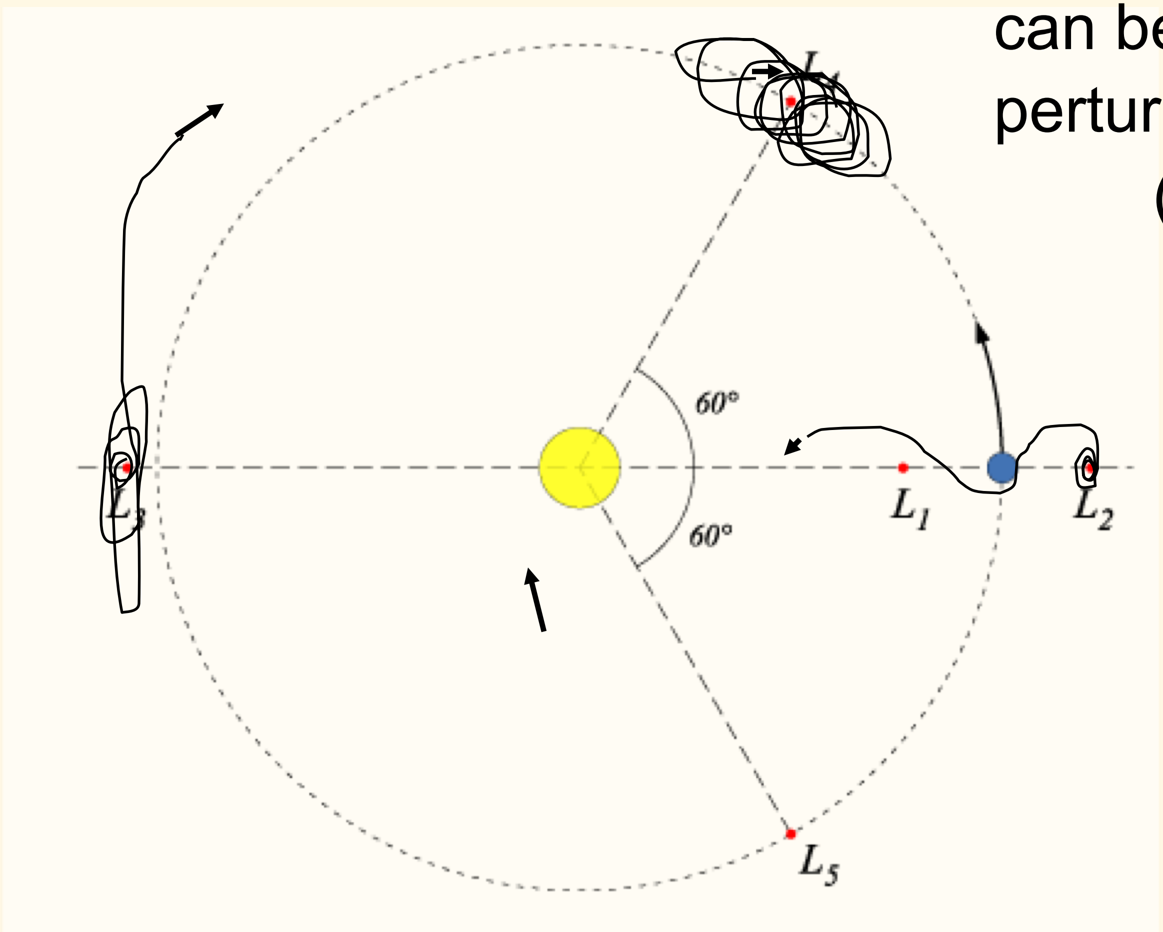
Is the motion around Lagrange points stable?

Stable could mean many things.

Linear stability requires that equilibrium is stable against infinitesimal perturbations.

Here, we'll talk about Liapunov stability which is only slightly different : a particle does not depart beyond a certain small radius at any (even infinite) time. It does not need to tend toward an equilibrium point, just not to depart from it much.

Is the motion around Lagrange points stable?



Stability of motion near L-points can be studied in the 1st order perturbation theory

(with unperturbed motion being state of rest at equilibrium point).

Stability of Lagrange points

Although the **L_1, L_2, and L_3 points** are nominally **unstable**, it turns out that it is possible to find **stable and nearly-stable periodic orbits** around these points in the R3B problem. They are used in the Sun-Earth and Earth-Moon systems for space missions parked in the vicinity of these L-points.

By contrast, despite being the maxima of effective potential, **L_4 and L_5 are stable equilibria**, provided M_1/M_2 is > 24.96 (as in Sun-Earth, Sun-Jupiter, and Earth-Moon cases). When a body at these points is perturbed, it moves away from the point, but the Coriolis force then bends the trajectory into a stable orbit around the point.

The strange thing is, $L_{4,5}$ are maxima of potential..

Observational proof of the stability of triangular equilibrium points

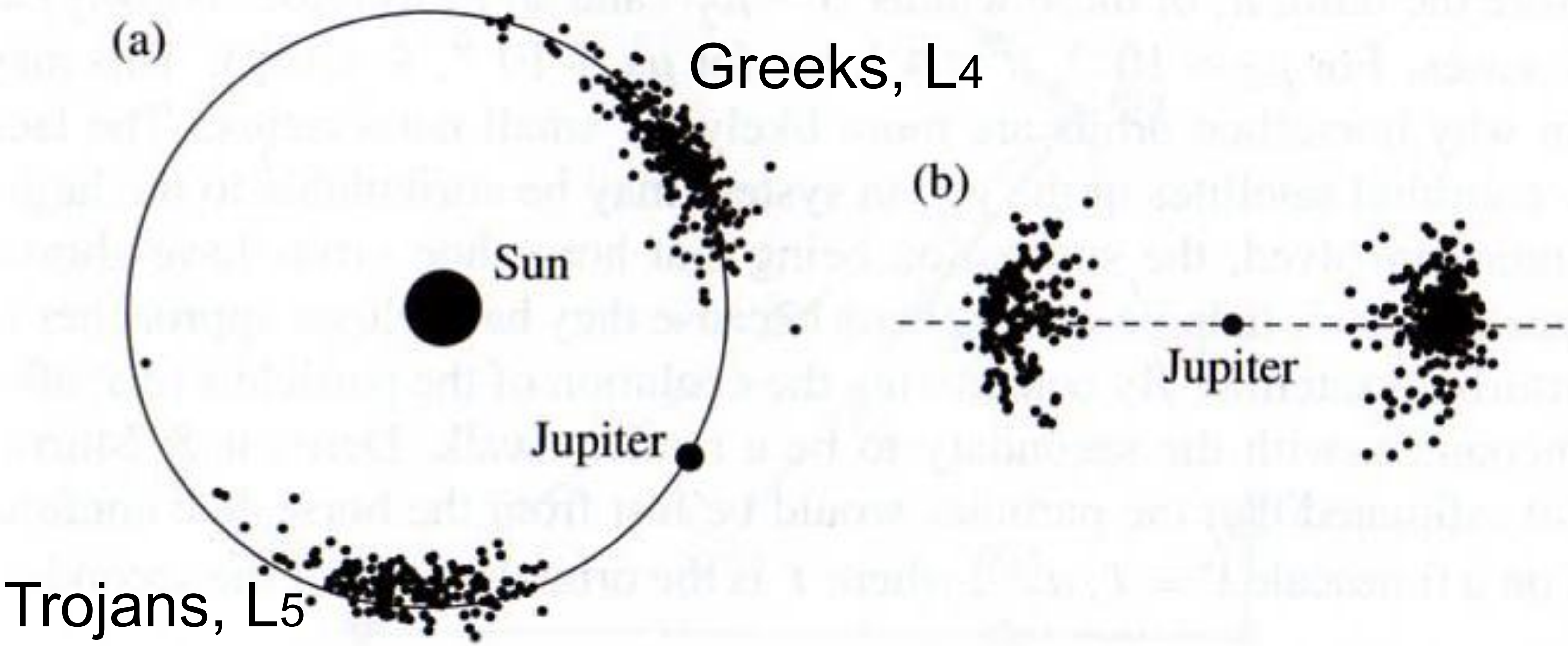
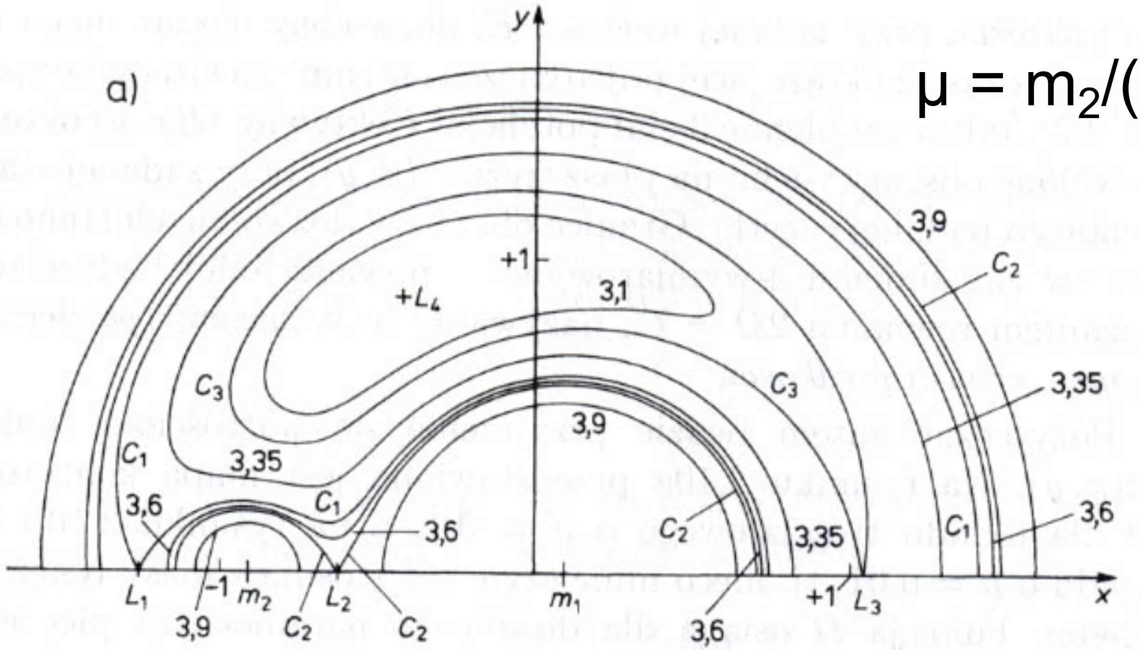


Fig. 3.23. (a) The distribution of asteroids in the vicinity of the orbit of Jupiter on December 18, 1997 at 0^h UT (Julian Date 2450800.5). The plot denotes the positions of the asteroids projected onto the plane of the ecliptic. (b) The vertical distribution of the same asteroids viewed along the Jupiter–Sun line. The dashed line denotes the plane of Jupiter's orbit.

From: *Solar System Dynamics,* by C.D. Murray and S.F.Dermott

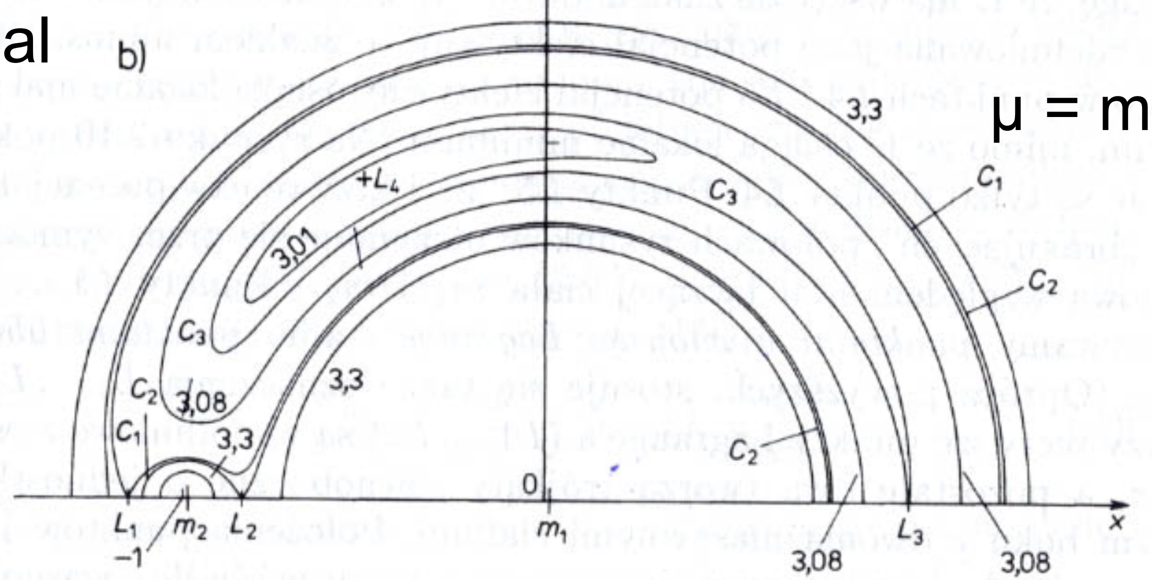
Roche lobe radius depends weakly on R3B mass parameter

$\mu = m_2/(m_1+m_2) = 0.1$



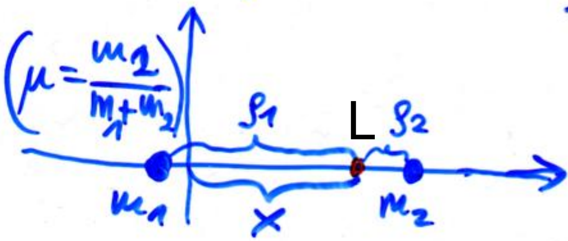
Lines are equipotential curves of the effective (gravity+centrif.) potential

$\mu = m_2/(m_1+m_2) = 0.01$



Computation of Roche lobe radius from R3B equations of motion ($r_L = \rho_2 a$, *a = semi-major axis of the binary, G=M=1)*

R3B: $\ddot{x} = x + 2\dot{y} + \left(+\frac{\mu}{\rho_2^2} - \frac{(1-\mu)}{\rho_1^2}\right)$



$0 = (\rho_1+\mu)\rho_1^2\rho_2^2 - \mu\rho_1^2 + (-1+\mu)\rho_2^2$

$x = \rho_1 - \mu$

$\rho_1 + \rho_2 = 1$

$\rho_2 =: r_L/a$ – 5th order equation

But for $\mu \ll 1$ simplifies to

$\frac{1}{\rho_1^2} = \frac{1}{(1-\rho_2)^2} \approx 1 + 2\rho_2$; $x = 1 - \rho_2 - \mu$

$0 = (1 - \rho_2 - \mu) + \frac{\mu}{\rho_2^2} - (1-\mu)(1+2\rho_2)$

$0 = -3\rho_2 + \frac{\mu}{\rho_2^2} + 2\mu\rho_2 \approx \frac{\mu}{\rho_2^2} - 3\rho_2$

$\Rightarrow \rho_2 = \left(\frac{\mu}{3}\right)^{\frac{1}{3}}$,

$$r_L \approx a\sqrt[3]{\frac{m}{3M}}$$

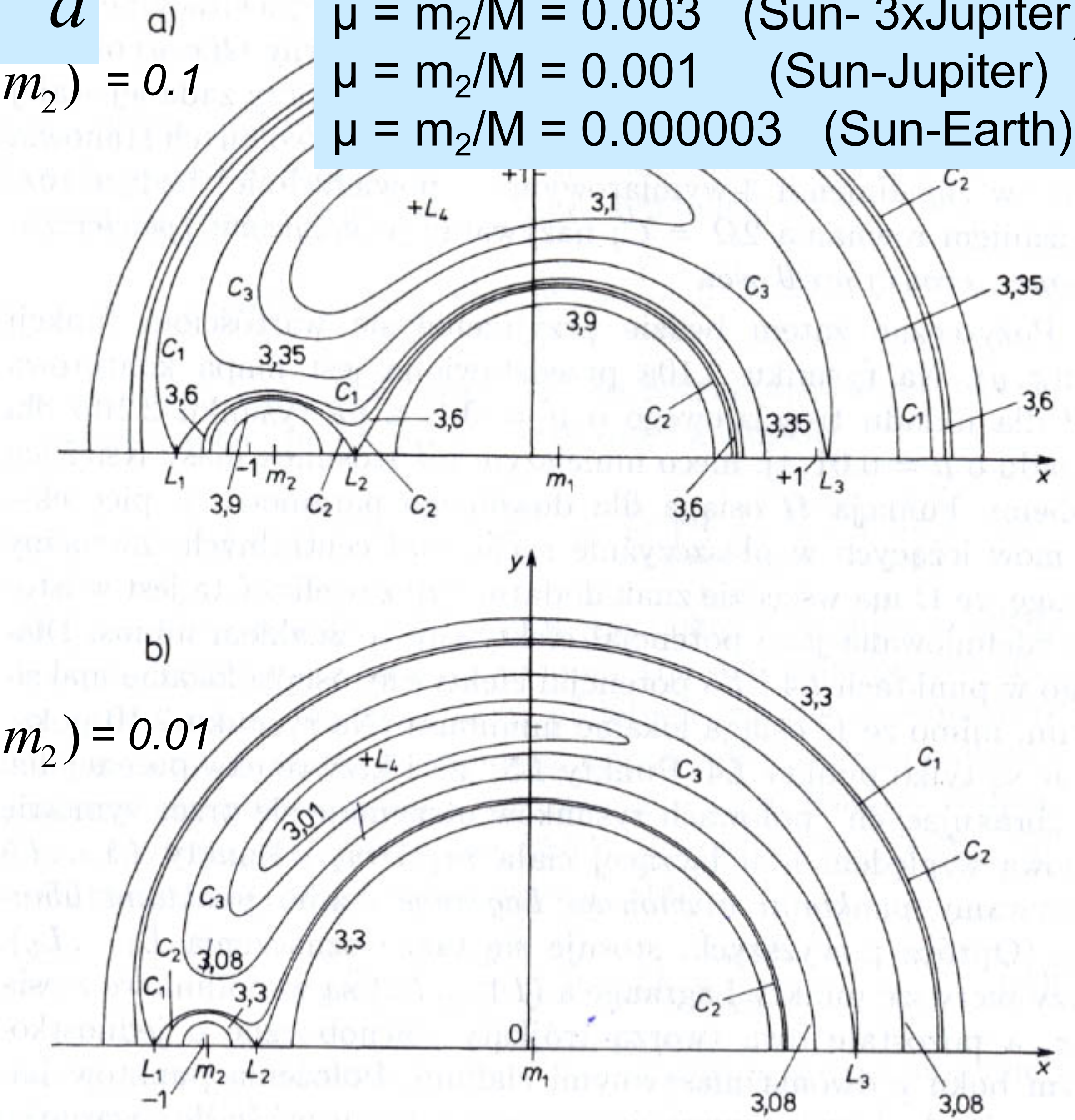
Roche lobe radius depends weakly on R3B mass parameter

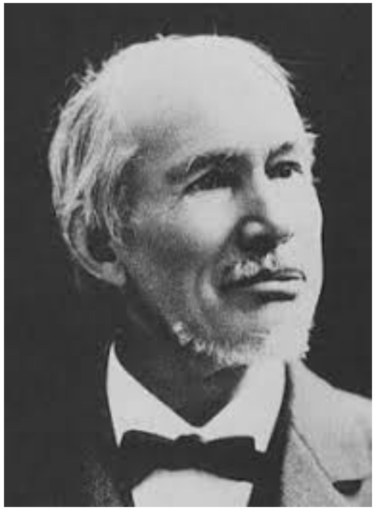
$$r_L = \left(\frac{\mu}{3}\right)^{1/3} a$$

μ = m_2/M = 0.01 (Earth ~Moon) r_L = 0.15 a
μ = m_2/M = 0.003 (Sun- 3xJupiter) r_L = 0.10 a
μ = m_2/M = 0.001 (Sun-Jupiter) r_L = 0.07 a
μ = m_2/M = 0.000003 (Sun-Earth) r_L = 0.01 a

$\mu = m_1/(m_1+m_2)$ = *0.1*

$\mu = m_1/(m_1+m_2)$ = *0.01*





Hill's problem

A simplification of Roche problem or
Circular Restricted 3-Body problem
for $\mu = m_2/(m_1+m_2) << 1$

George W. Hill (1838-1914)

Received M.A. from Rutgers U.; loved living with his 8 siblings in West Nyat, NY. Worked at Columbia U. and in Washington in Naval Office but hated the place. Pioneer of work-from-home ☺

Hill studied the small mass ratio limit in local Cartesian coordinates attached to the planet (mass m_2 in general).
He 'straightened' the azimuthal coordinate by replacing it with a local Cartesian coordinate *y*, and replaced radial coordinate *r* with *x*. The problem can be written as 2D or 3D (we do 3D below).

.

$\ddot{x} = -(\nabla\varphi)_x + \Omega^2 x + 2\Omega\, y$ *Eqs. of motion of Hill*

$\ddot{y} = -(\nabla\varphi)_y + \Omega^2 y - 2\Omega\, \dot{x}$ *in a frame rotating*

$\ddot{z} = -(\nabla\varphi)_z$ *at ang. speed* Ω

Ω is the mean motion (Ω == n) of the binary system of masses, and time derivatives are denoted by $\dot{x}$ =dx/dt (x-velocity) and $\ddot{x}$ (x-acceleration), etc.

$f = -\nabla\varphi$ stands for the linearized gravitational force field (per unit mass of the test particle) due to bodies 1 and 2. Hill's eqs. are valid locally around m_2 body, e.g. x,y,z are relative to planets position and all << a.

$\ddot{x} = -\mu GM\, x/r^3 + 3\Omega^2 x + 2\Omega\, \dot{y}$

$\ddot{y} = -\mu GM\, y/r^3 \qquad - 2\Omega\, \dot{x}$

$\ddot{z} = -\mu GM\, z/r^3 \qquad$ *where* $r^2=x^2+y^2+z^2 << a^2$

Hill's eqs. are valid locally around the smaller body.
Let's use $Gm_2/r^3 = \mu GM/r^3 = \mu\Omega^2(a/r)^3$ (since $GM/a^3=\Omega^2$) and a definition of Roche lobe as a characteristic, small distance defining the range of planet's or secondary star's gravitational influence

$r_L = a\ (\mu/3)^{1/3}$ ➔ $r_L^{\ 3} = a^3\ \mu/3$ ➔ $a^3\ \mu = 3\ r_L^{\ 3}$

Then $\mu\Omega^2(a/r)^3 = 3\Omega^2\ (r_L/r)^3$.

Changing the definition from dimensional x,y,z to nondimensional ratios $x = x/r_L$ $y = y/r_L$ etc., we write

$$\ddot{x} = -\ 3\Omega^2\ (x/r^3 - x) + 2\Omega\ dy/dt$$
$$\ddot{y} = -\ 3\Omega^2\ y/r^3 \qquad -2\Omega\ dx/dt$$
$$\ddot{z} = -\ 3\Omega^2\ z/r^3,$$

where $r = r/r_L = (x^2+y^2+z^2)^{1/2}$

Hill's non-dimensional equations can further be simplified by introducing non-dimensional *time* $t = \Omega$ t

$x'' = -3\,x\,(r^{-3} - 1) + 2\,y'$

$y'' = -3\,y/r^3 \qquad -2\,x'$ where $' = d/dt$, $'' = d^2/dt^2$

$z'' = -3\,z/r^3$

We can immediately see that the 2 Lagrange points in Hill's equations are at

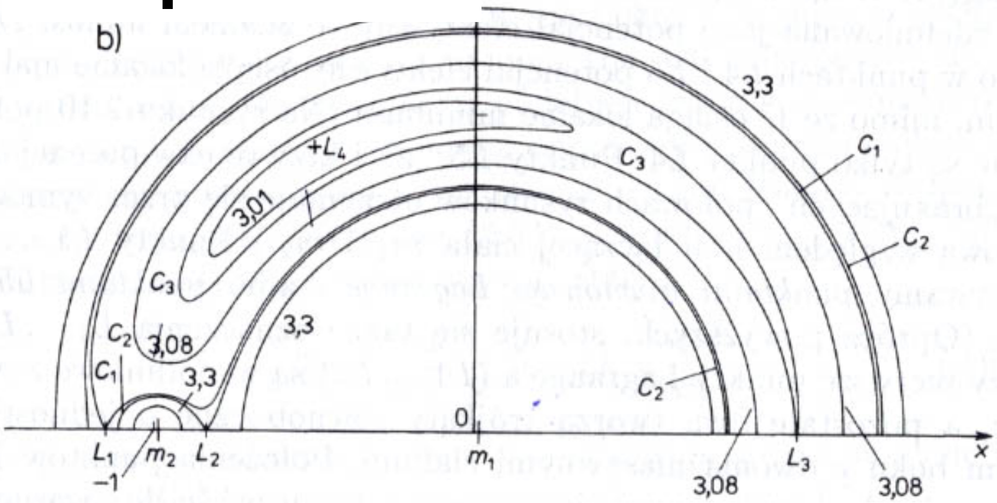
$x = \pm 1,\ y=0,\ z=0$ (at $r = 1$).

There, all second time derivatives (accelerations) vanish, if velocities $x' = dx/dt$ and $y' = dy/dt$ vanish.

These two locations are thus equilibrium points.

The triangular L points are not there: they're much outside the radius of validity of the Hill's local equations, and only exist in the circular, non-local R3B.

Hill problem



Here, you see the straightening of the curve-linear equipotential lines of the full and CR3B problem in the local Hill coordinates. The lower figure is in fact valid for any mass ratio μ, as long as μ is small, everything scales with Roche lobe size r_L.

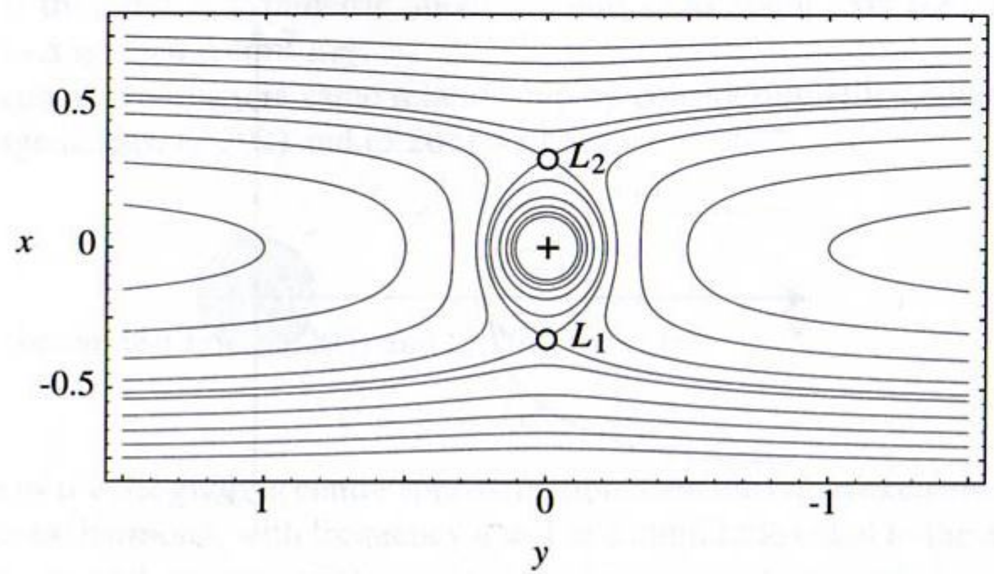
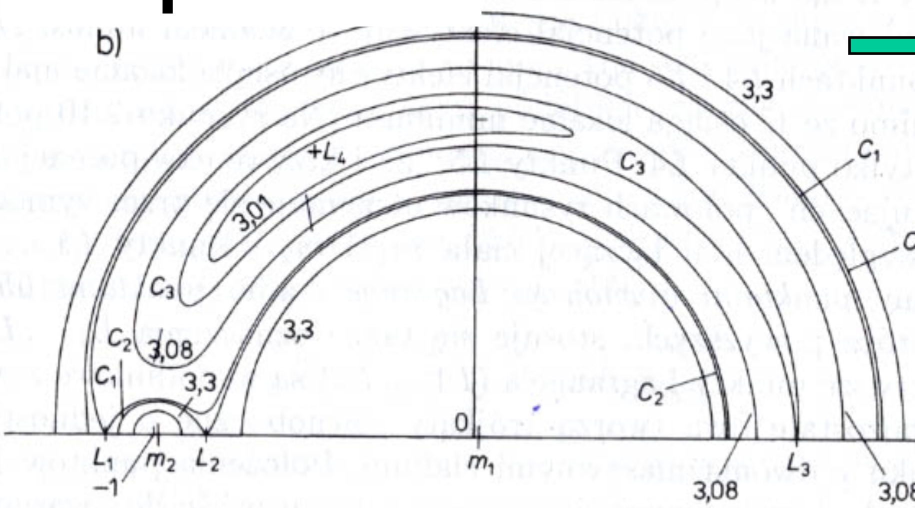


Fig. 3.28. The zero-velocity curves defined by the equation $C_H = 2U_H$ in the vicinity of the Lagrangian points L_1 and L_2 for a mass $\mu_2 = 0.1$. Note that in the Hill's approximation the equilibrium points are now equidistant from the mass μ_2 (denoted by the cross at the origin).

In particular, the distance from L_1 to L_2 becomes $2r_L$.

Hill problem



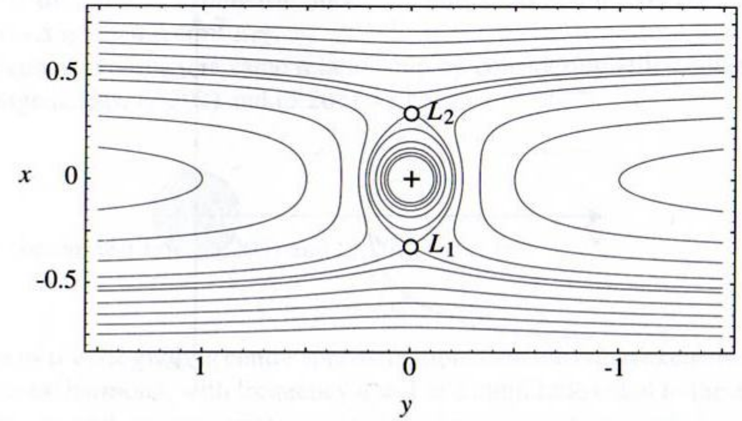


Fig. 3.28. The zero-velocity curves defined by the equation $C_H = 2U_H$ in the vicinity of the Lagrangian points L_1 and L_2 for a mass $\mu_2 = 0.1$. Note that in the Hill's approximation the equilibrium points are now equidistant from the mass μ_2 (denoted by the cross at the origin).

$$r_L \mapsto r_L = \left(\frac{\mu}{3}\right)^{1/3} a$$

G.W. Hill applied his equations to the Sun-Earth-Moon problem, showing that the Moon's Jacobi constant C=3.0012 is larger than C_L=3.0009 (value of effective potential at the L-point), which means that its Zero Velocity Surface lies inside its Hill sphere and no escape from the Earth is possible: the Moon is Hill-stable.

However, this is not a strict proof of Moon's eternal stability because:

(1) Circular orbit of the Earth was assumed (crucial for constancy of Jacobi's C)

(2) Moon was approximated as a massless body, like in R3B.

(3) Energy constraints can never exclude the possibility of Moon-Earth collision

COMPARISON OF DIFFERENT THEORIES WE'VE LEARNED

From the example of Sun-Earth-Moon system we find that:

- Classical Lagrange-Laplace **perturbation theory** often has non-convergent time series, useful for limited time only. Analytical methods of Laplace and Lagrange were OK in their time, when the biblical age of the Sun/Earth of 4000 yr was accepted.
- **Integrals of motion** guarantee no-escape from the allowed regions of motion for an *infinite* period of time, which is better than either the **general** or the **special perturbation theory** but only if the assumptions of the theory are satisfied, and that's difficult to achieve in practice
- We are usually interested in time periods up to Hubble time or more. In late 1990s our computers and algorithms became capable of simulating such enormous time spans. Thus numerical exploration has supplanted the elegant 18th-century methods and is the preferred tool of a dynamicist trying to ascertain the stability of the Solar System and its exo-cousins.

Is the Solar System orbitally stable?

Yes, it appears so in practical sense (no orbit crossings, ejections, collisions of major bodies for billions of years), but we cannot be absolutely sure!

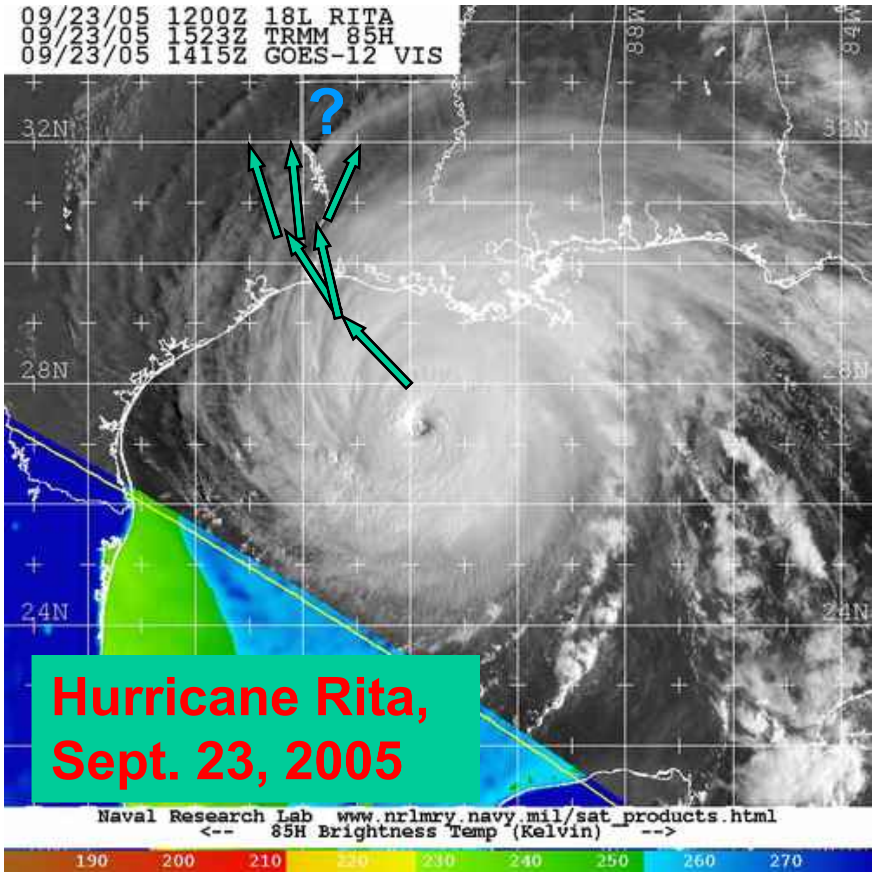
Semi-analytical and numerical simulations of the future of Solar System show that **chaos** rules the orbits on long enough time scales. Beyond a certain time (called Lyapunov time), results become a statistics of various possible outcomes rather than a unique prediction.

Chaos does not necessarily mean that *orbits are crossing* or that there must come to a mayhem. The more massive planets are always near their current places on timescale of Hubble time (10 Gyr).
It may mean that we don't know exactly the orientation and eccentricity of an orbit, and the position along that elliptic path.

So is the Solar System stable for sure?

There is no certainty, now or ever.

The reason is that, like the weather on Earth, the detailed configuration of the planets after 1 Gyr, or even 100 mln yr is impossible to predict or compute.



Hurricane Rita, Sept. 23, 2005

(it weakened unexpectedly fast)

On Earth, this is because of chaos in weather systems (super-sensitivity to initial conditions, too many coupled variables)

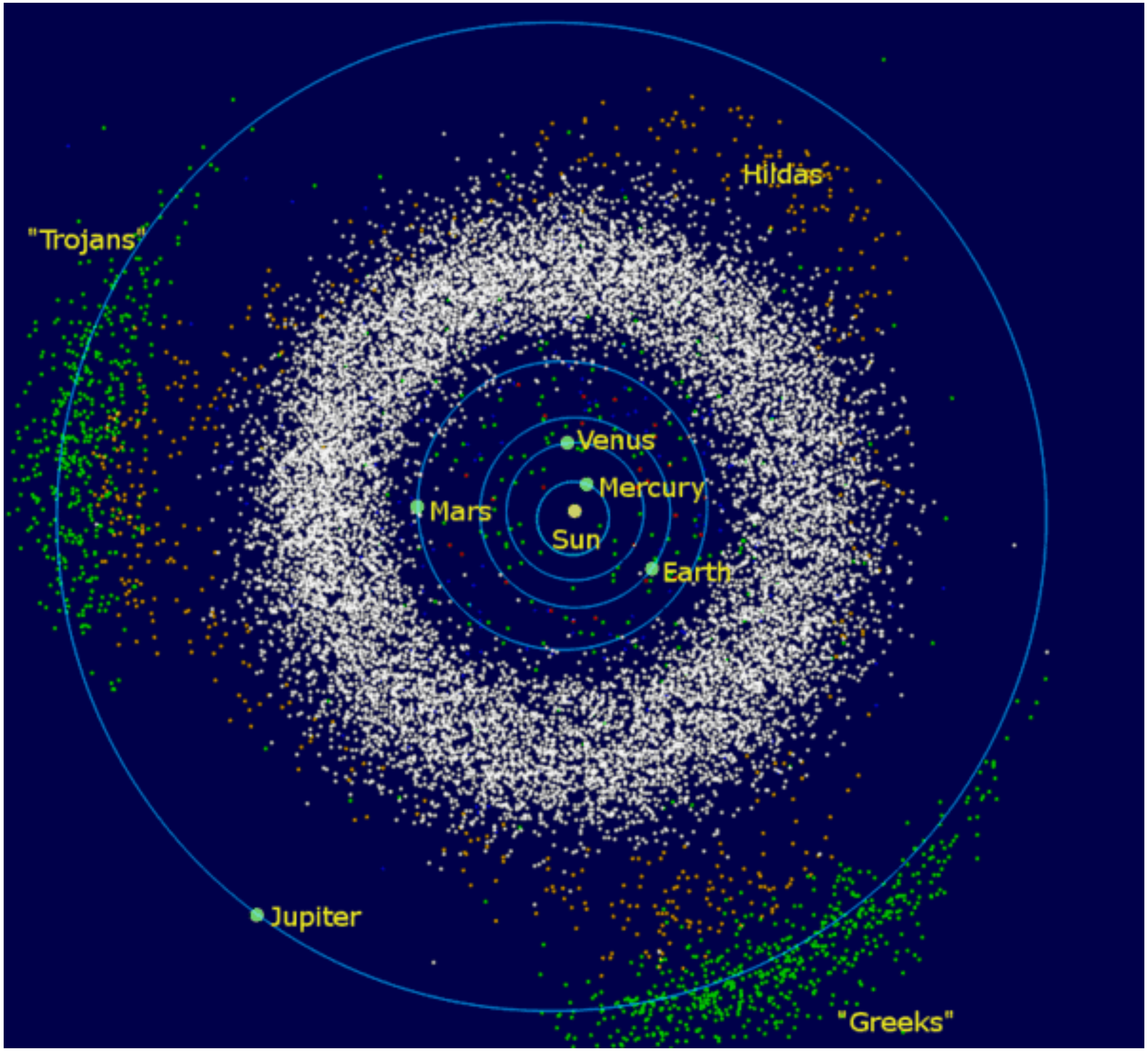
In planetary systems, chaos is due to planet-planet gravitational perturbations amplified by resonances. Two or more overlapping resonances can make the precise predictions of the future futile.

Asteroid Mathilde, photographed by spacecraft in 1997.

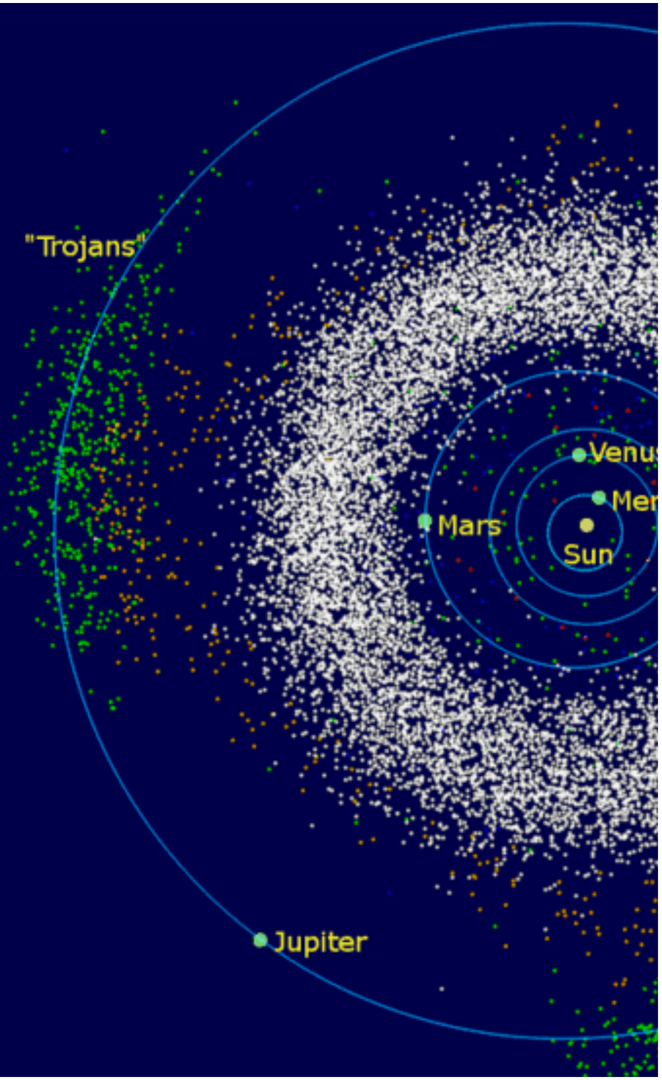
Size: 59 x 47 km

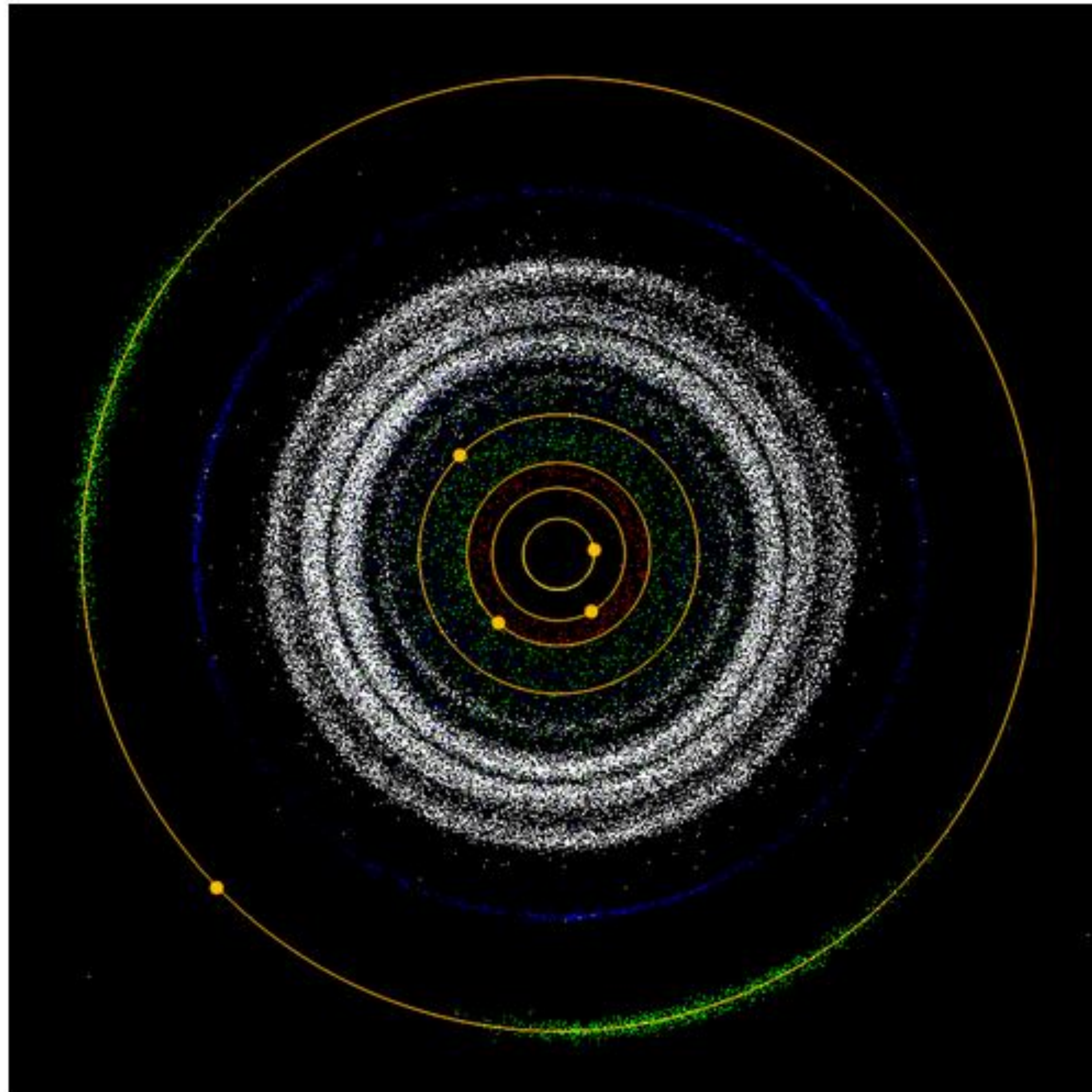
Rotates chaotically

ORBITAL RESONANCES - example: Astroid Belt between Mars and Jupiter. Clearly visible are 1:1 resonant objects (Trojans and Greeks). Other commensurabilities of mean motions (periods) *are* present but smeared out by eccentric motion on elliptical orbits.

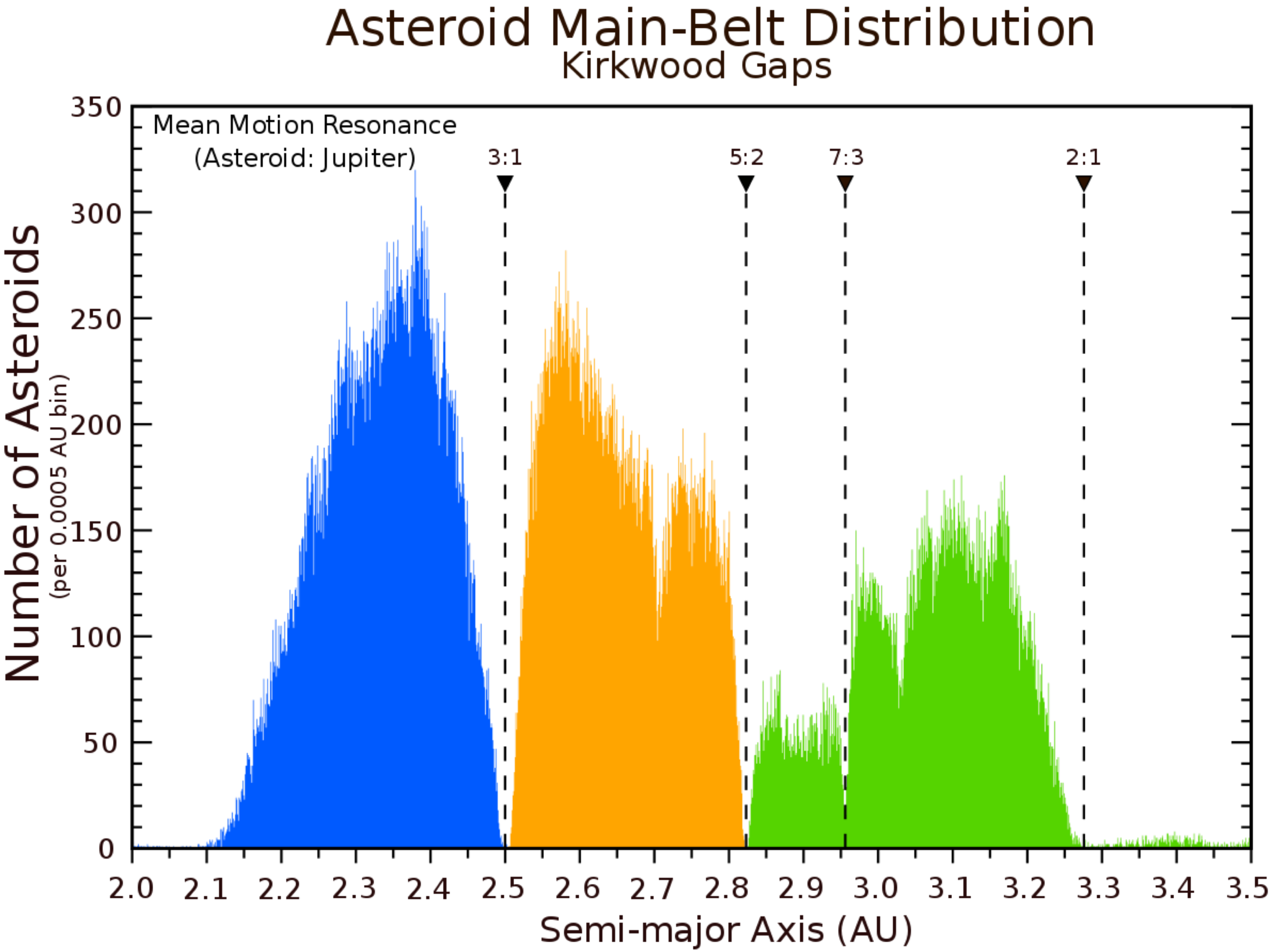


ORBITAL RESONANCES – visualization of a and ω as polar coord. on the right.

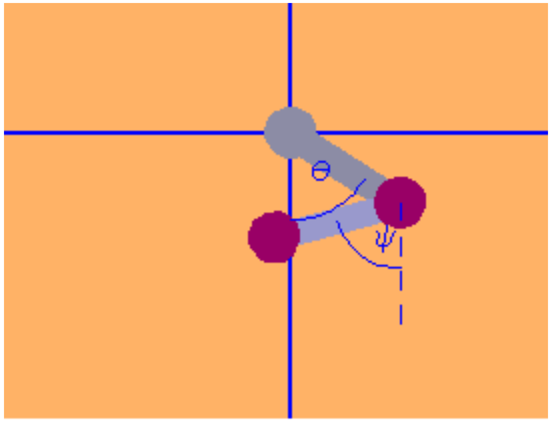




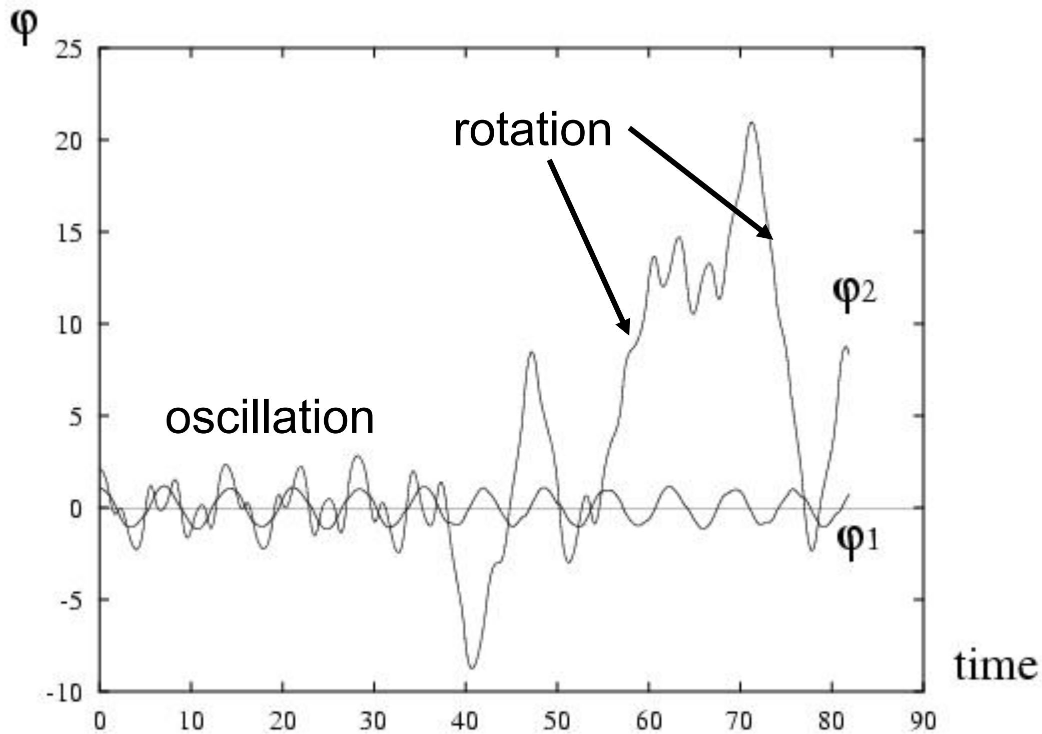
Resonances are of different types, e.g., mean-motion commensurabilities that we find in the so-called Kirkwood gaps:



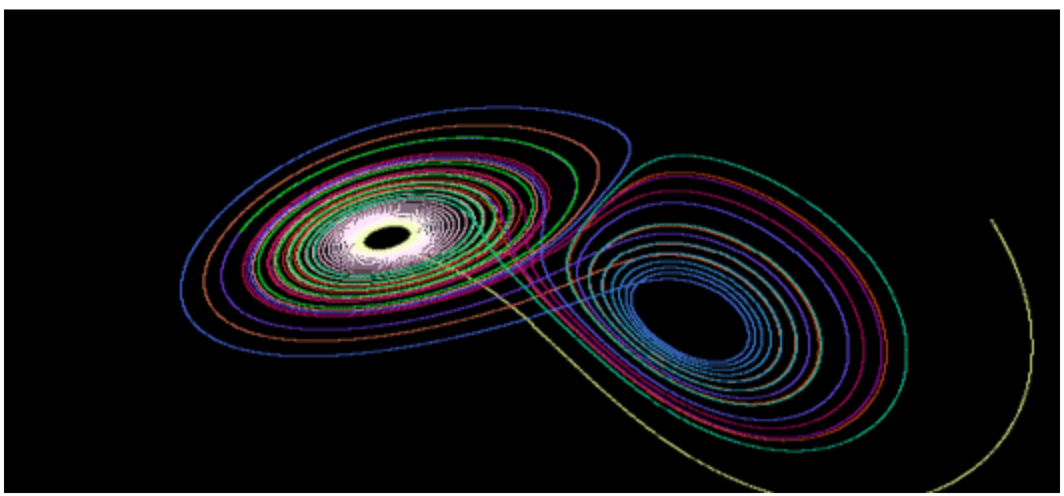
Chaos in:



Double pendulum



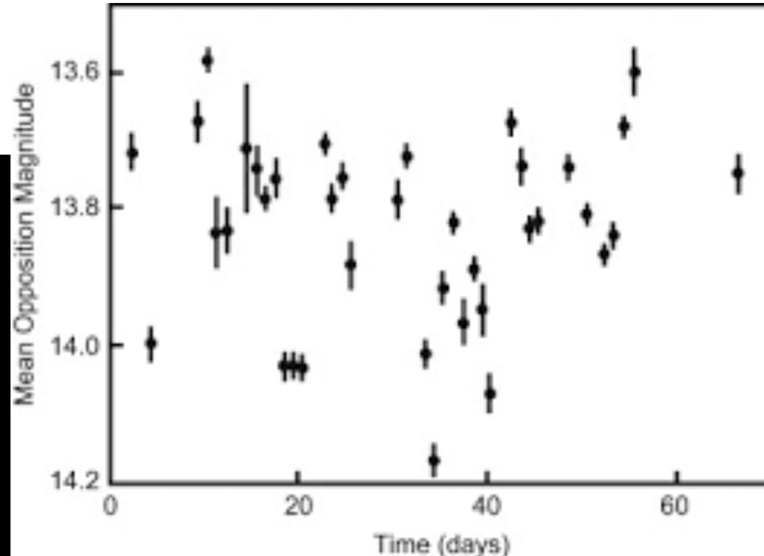
$$\frac{dx}{dt} = -10x + 10y, \quad \frac{dy}{dt} = 28x - y - xz, \quad \frac{dz}{dt} = -\frac{8}{3}z + xy$$

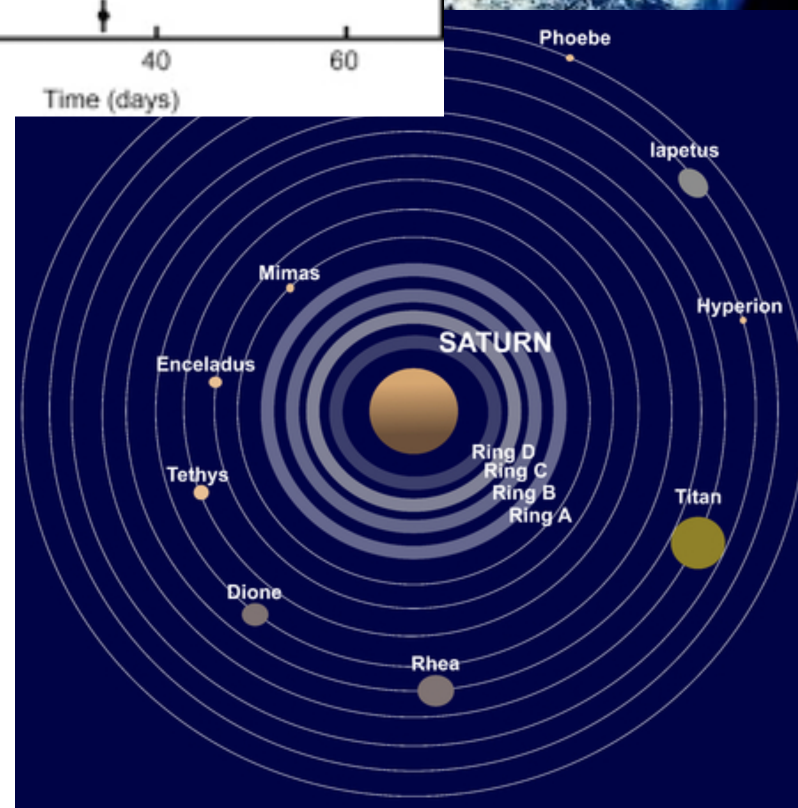


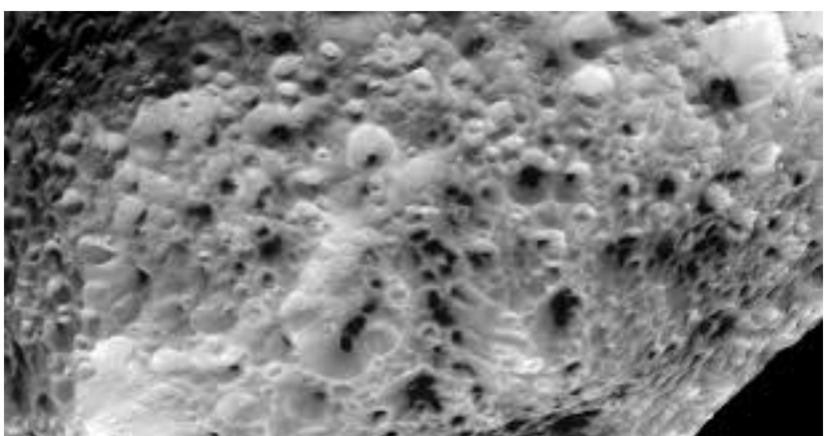
Lorentz attractor
(modeled after weather system equations by meteorologist Ed Lorenz)

Lorenz attractor

Hyperion a Saturnian satellite, the only satellite showing chaotic **rotation** (light curve is aperiodic)







In the Solar System, in 2-body resonances resonant angles librate (i.e. oscillate)

Table 8.8. Known first- and second-order mean motion resonances involving planets or satellites in the solar system. In each case the unprimed and primed quantities refer to the inner and outer bodies respectively. All known planetary and satellite resonances are included.

System	Resonant Argument	Amplitude	Period (y)
Planets			
Neptune–Pluto	$3\lambda' - 2\lambda - \varpi'$	76°	19,670
Jupiter			
Io–Europa	$2\lambda' - \lambda - \varpi$	1°	—
Io–Europa	$2\lambda' - \lambda - \varpi'$	3°	—
Europa–Ganymede	$2\lambda' - \lambda - \varpi$	3°	—
Saturn			
Mimas–Tethys	$4\lambda' - 2\lambda - \Omega' - \Omega$	43.6°	71.8
Enceladus–Dione	$2\lambda' - \lambda - \varpi$	0.297°	11.1
Titan–Hyperion	$4\lambda' - 3\lambda - \varpi'$	36.0°	1.75

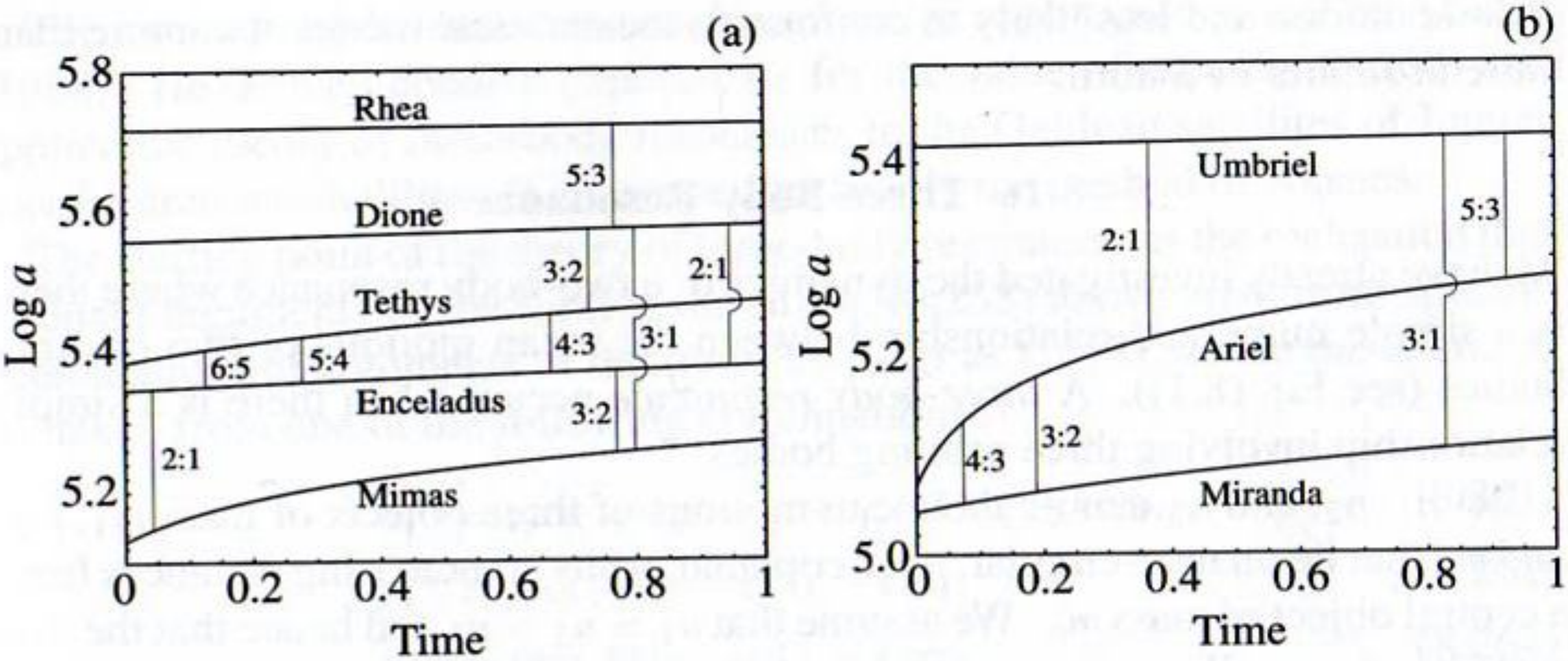


Fig. 8.29. Sample changes in the semi-major axes (measured in km) of satellites in (a) the saturnian and (b) the uranian system. A selection of first- and second-order resonances between pairs of satellites is indicated on each plot.

Strong, non-chaotic resonances are present in satellite systems.
Also the planets exhibit such low-order (near) commensurabilities, the most famous being the 2:5 Saturn-Jupiter one. (2:3 Pluto-Neptune resonance does not prevent the chaotic nature of Pluto's orbit.)

Example of a chaotic orbit due to *overlapping* resonances

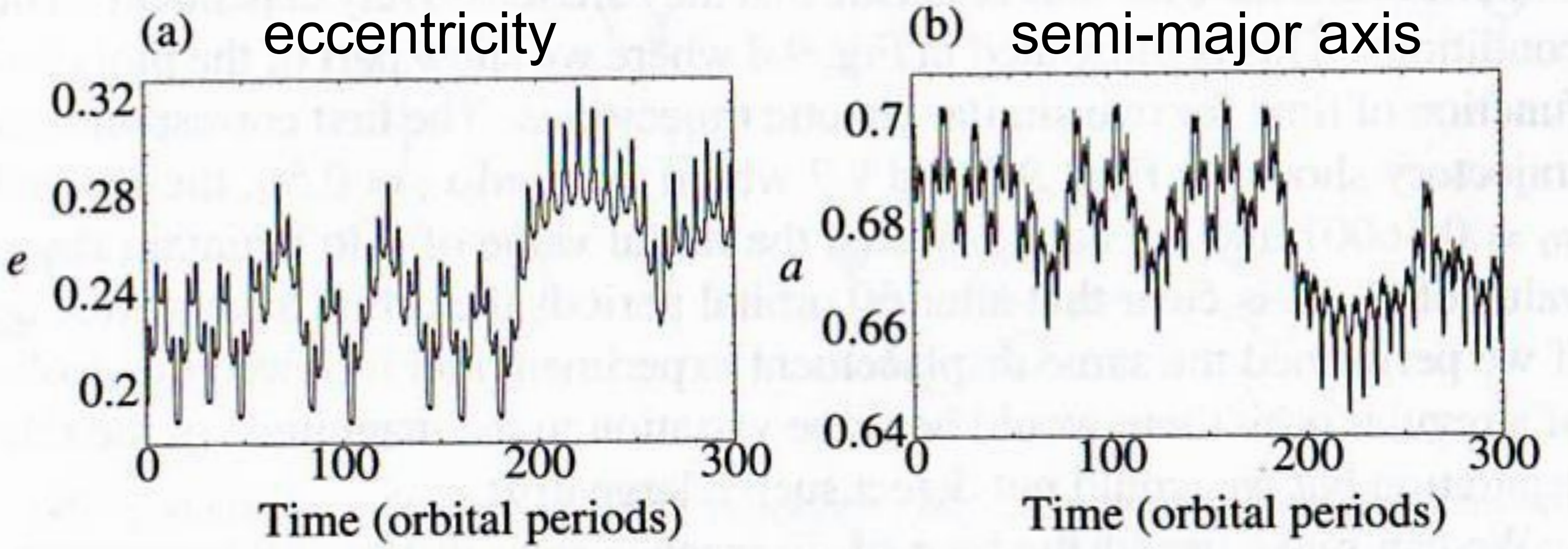


Fig. 9.6. The time variability of (a) eccentricity e and (b) semi-major axis a for initial values $a_0 = 0.6984$ and $e_0 = 0.1967$. The plots show a behaviour characteristic of chaotic orbits. (Adapted from Murray 1998.)

Orbits and planet positions on them are unpredictable on a timescale of 100 mln yr or less (50 mln yr for Earth).

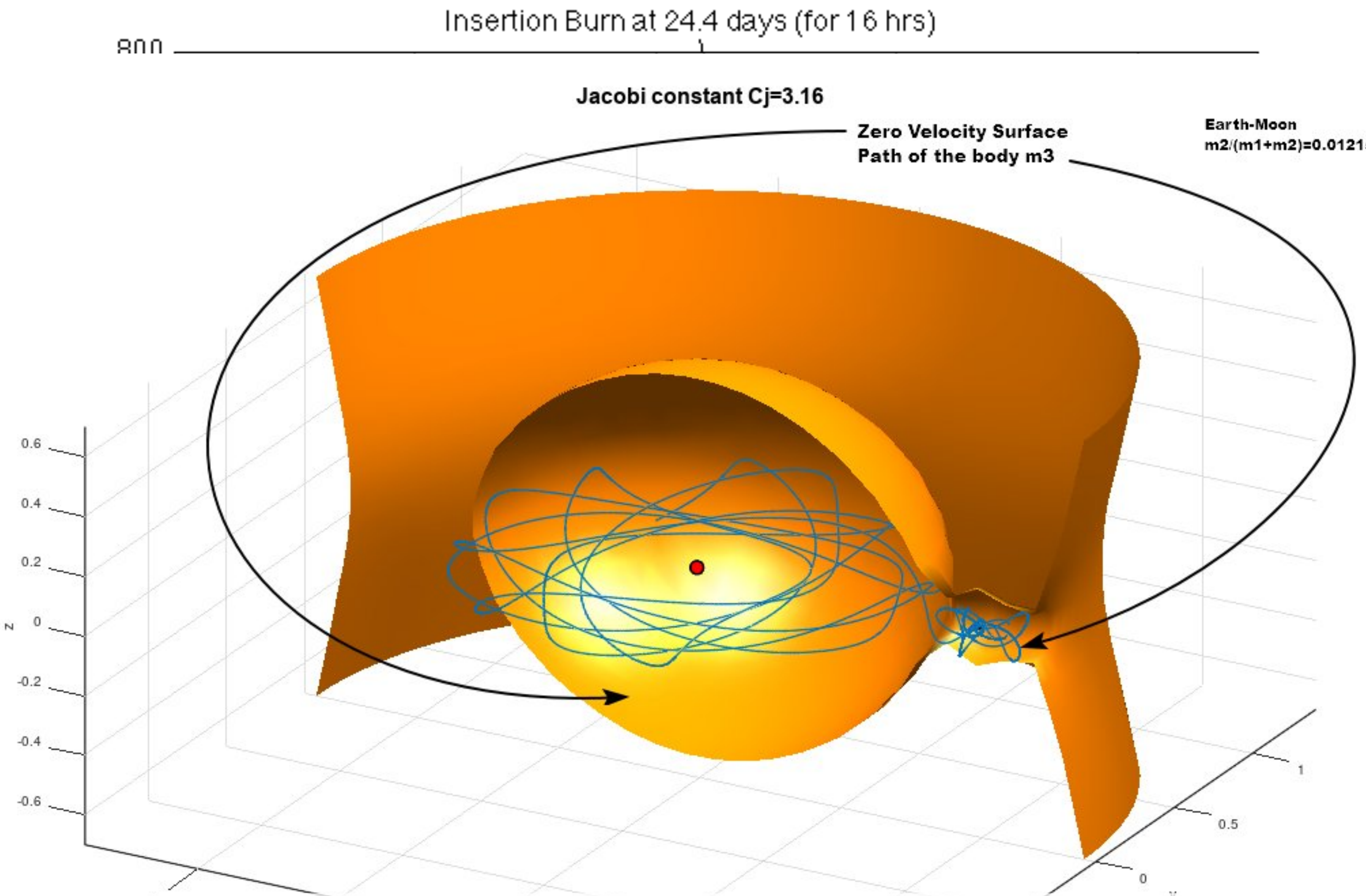
For instance, let the longitudes of perihelia be denoted by ω and the ascending nodes as Ω, then using subscripts E and M for Earth and Mars, there exists a resonant angle

$$f_{ME} = 2(\omega_M - \omega_E) - (\Omega_M - \Omega_E)$$

that shows the same hesitating behavior between oscillation (libration) and circulation (when resonant lock is broken) as in a double pendulum experiment.

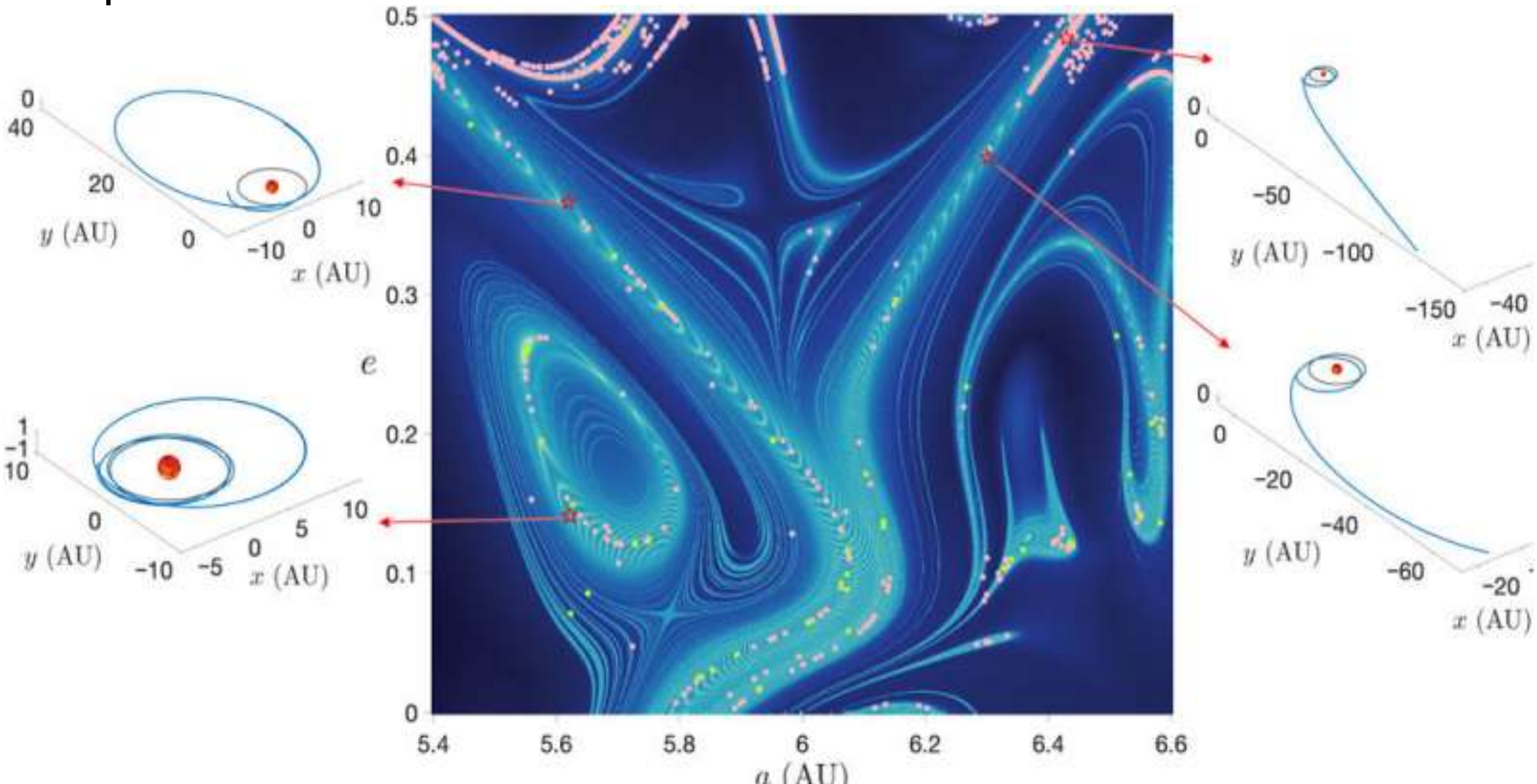
But **chaos** in our system is long-term **stable** for a time of order of its age. Orbits have the numerical, long-term, stability. They don’t cross and planets don’t exchange places or get ejected into Galaxy.

The only questionable stability case is that of Mercury & Sun. Under the action of more massive planets, Mercury makes such wide excursions in orbital elements that in some simulations it drops onto the Sun in 3-10 Gyr.



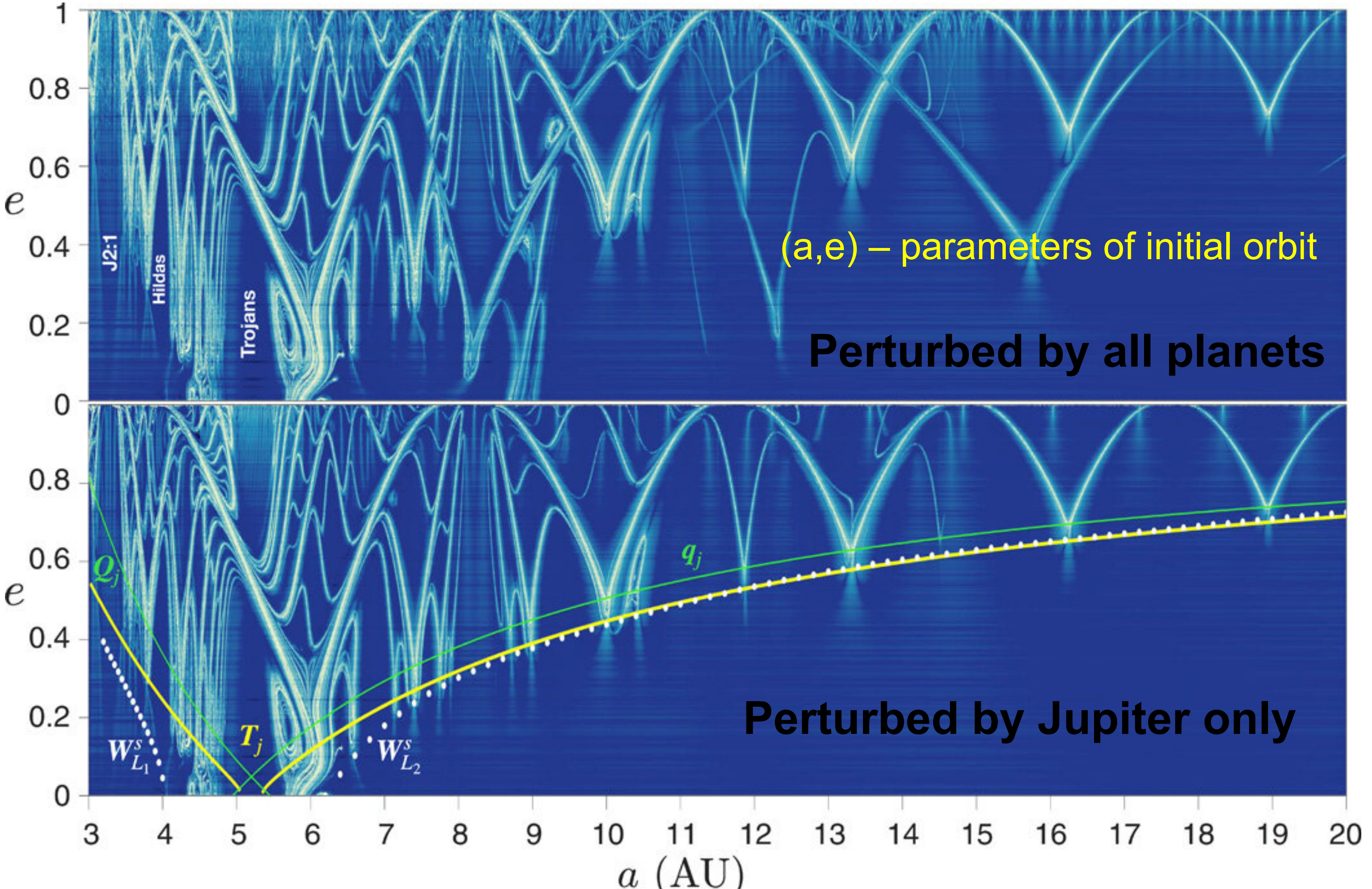
Trajectories are often computable fairly precisely for small number of orbits only. On long time scales they are chaotic. Re-entry into the Roche lobe of a planet can occur occasionally.

There are indicators of chaos (so-called fast Lyapunov exponents) that can map stable and unstable manifolds in parameter space *(a,e)* for asteroids like Centaurs (in Jupiter-Neptune region). Centaurs use overlapping resonances to travel fast (in a few million yr) from the outer to the inner Solar System. Bright stripes are like highways for rapid transport ofminor bodies



Nataša Todorović et al. *The arches of chaos in the Solar System*, Science Adv. (2020).

Lighter = more chaotic (confusingly called 'stable manifolds' in the paper)



https://phys.org/news/2020-12-accessing-arches-chaos-solar-fast.html

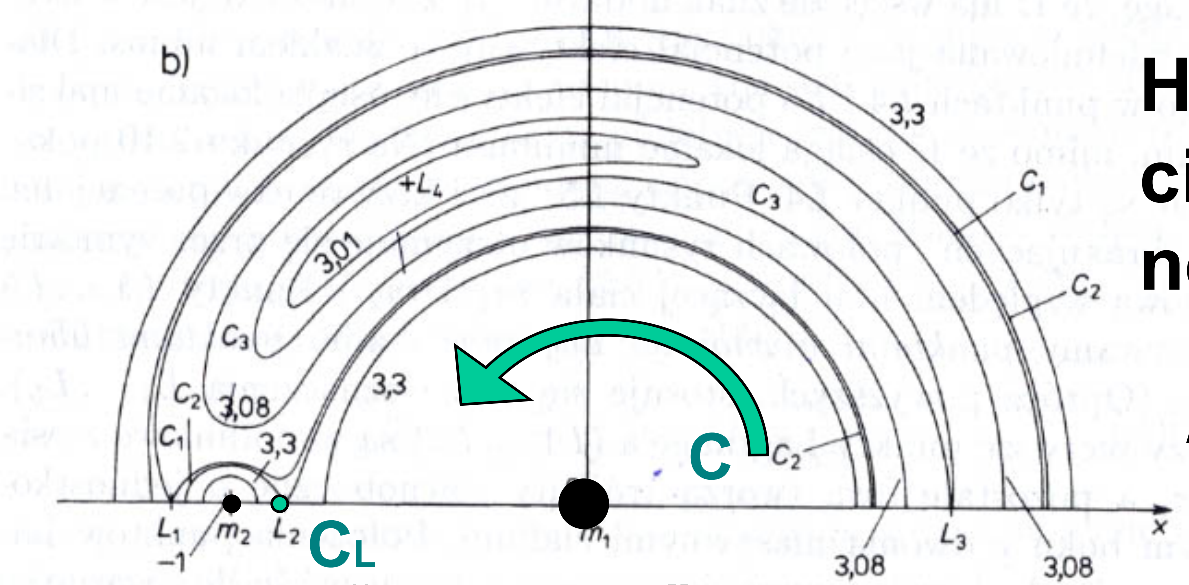
How wide a region is quickly destabilized by a planet?

(We call it Corotational Region)

The gravitational influence of a small body (a planet around a star, for instance) dominates the motion inside its Roche lobe, so particle orbits there are circling around the planet, not the star. The circumstellar orbits (usually in a disk), in the vicinity of the planet's orbit are affected, too.

To what radial extent?

Corotational region defines the 'feeding zone' of a growing protoplanet. We will see how it is populated by tadpoles and littered by horseshoes...



Hill stability of circumstellar motion near the planet

$$r_L \mapsto r_L = \left(\frac{\mu}{3}\right)^{1/3} a$$

Bodies on "disk orbits" (meaning the disk of bodies circling around the star) have Jacobi constants C depending on the orbital separation parameter x = (r-a)/a (r=initial circular orbit radius far from the planet, a = planet's orbital radius). If |x| is large enough, the disk orbits are forbidden from approaching L1 and L2 and entering the Roche lobe by energy constraint. Their effective energy is not enough to pass through the saddle point of the effective potential.

Disk regions farther away than some minimum separation |x| (assuming circular initial orbits) are guaranteed to be Hill-stable, or isolated from the planet by Jacobi energy constrain.

It is easy to compute the marginal orbital spacing in the Hill problem. In vector form $\mathbf{v} = d\mathbf{r}/dt$, and the Hills equations read, using unit *vector* $\mathbf{\Omega}$ pointing in the vertical direction

$$d^2\mathbf{r}/dt^2 = -\nabla\Phi + 2(\mathbf{\Omega} \times \mathbf{v}) = -\nabla(\text{gravity \& tidal pot.}) + \text{Coriolis force}$$

where the effective potential Φ of combined planet's gravity and sun's tidal force reads: $\Phi = -3(1/r + x^2/2).$

Taking a dot product with **v** of both sides, and using on r.h.s.

$(\mathbf{\Omega} \times \mathbf{v}) \cdot \mathbf{v} = 0$, we obtain Jacobi energy integral

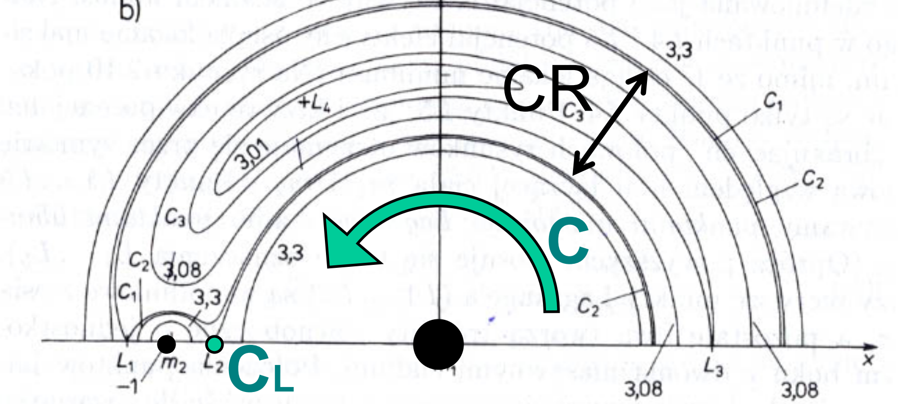
$$E_J = v^2/2 + \Phi = const.$$

- Its value for a particle at rest in Lagrange point at $x = r = 1,\ y = 0$ (one Roche lobe radius from planet) is equal $E_L = \Phi = -9/2.$
- Very far from planet, at r=+∞, the Jacobi constant of a particle travelling with *x=const.* & asymptotic speed dy/dt= -(3/2)x, equals

$$E_J = +(3/2)^2\, x^2/2 \ -(3/2)x^2 = -(3/8)x^2.$$

From $E_J = E_L$ we obtain x (in units of Roche lobe radius r_L)

$x = (12)^{1/2} = 2\sqrt{3} \sim 3.5$ (Half-width of Corot. Region in units of r_L)



Hill stability in CR3B, of circumstellar motion near the planet's orbit

On a circular orbit with *x = (r-a)/a,*

$$C = 3 + \tfrac{3}{4}x^2$$

$$r_L = \left(\tfrac{\mu}{3}\right)^{1/3} a$$

At the L_1 and L_2 points

$$C_L = 3 + 9(r_L/a)^2$$

Therefore, the Hill stability criterion C = C_L reads

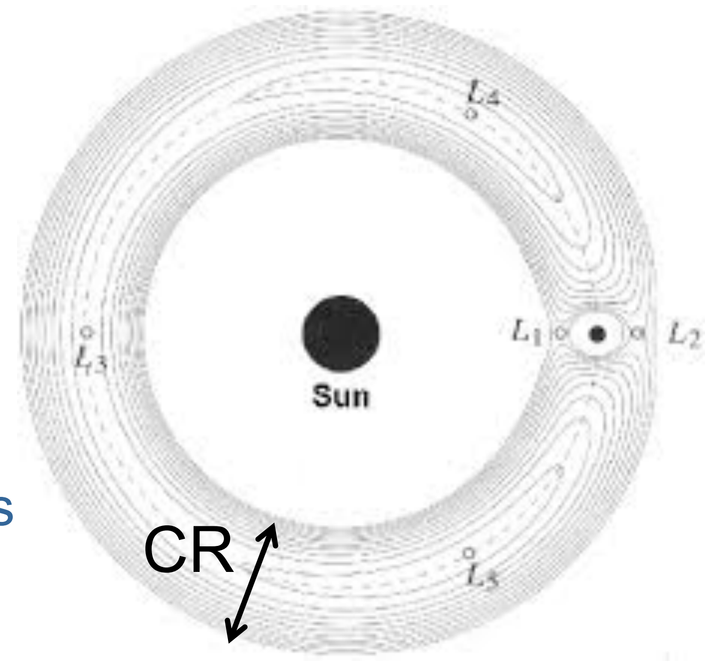
$$x^2 = 12(r_L/a)^2$$

or

$$x = 2\sqrt{3}(r_L/a) \approx 3.5\ r_L/a$$

Example: What is the radial extent of Hill-unstable region around Jupiter, also called its Corotational Region (CR)?

In this region we find tadpole and horseshoe orbits



Hill stability in CR3B, of heliocentric motion near planet Jupiter

$$x = 2\sqrt{3}(r_L / a) \approx 3.5 \ r_L / a \qquad r_L = (\tfrac{\mu}{3})^{1/3} a$$

What is the extent of Hill-unstable region around Jupiter (half-width)?

Jupiter-Sun mass ratio equals $\mu = 0.001$,

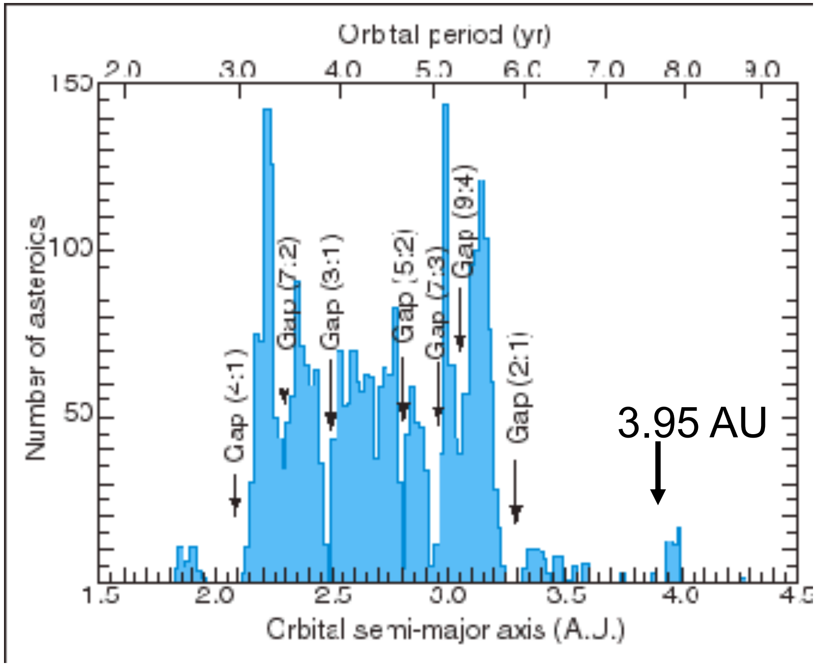
$x = 3.5 (\mu/3)^{1/3} = 0.24$

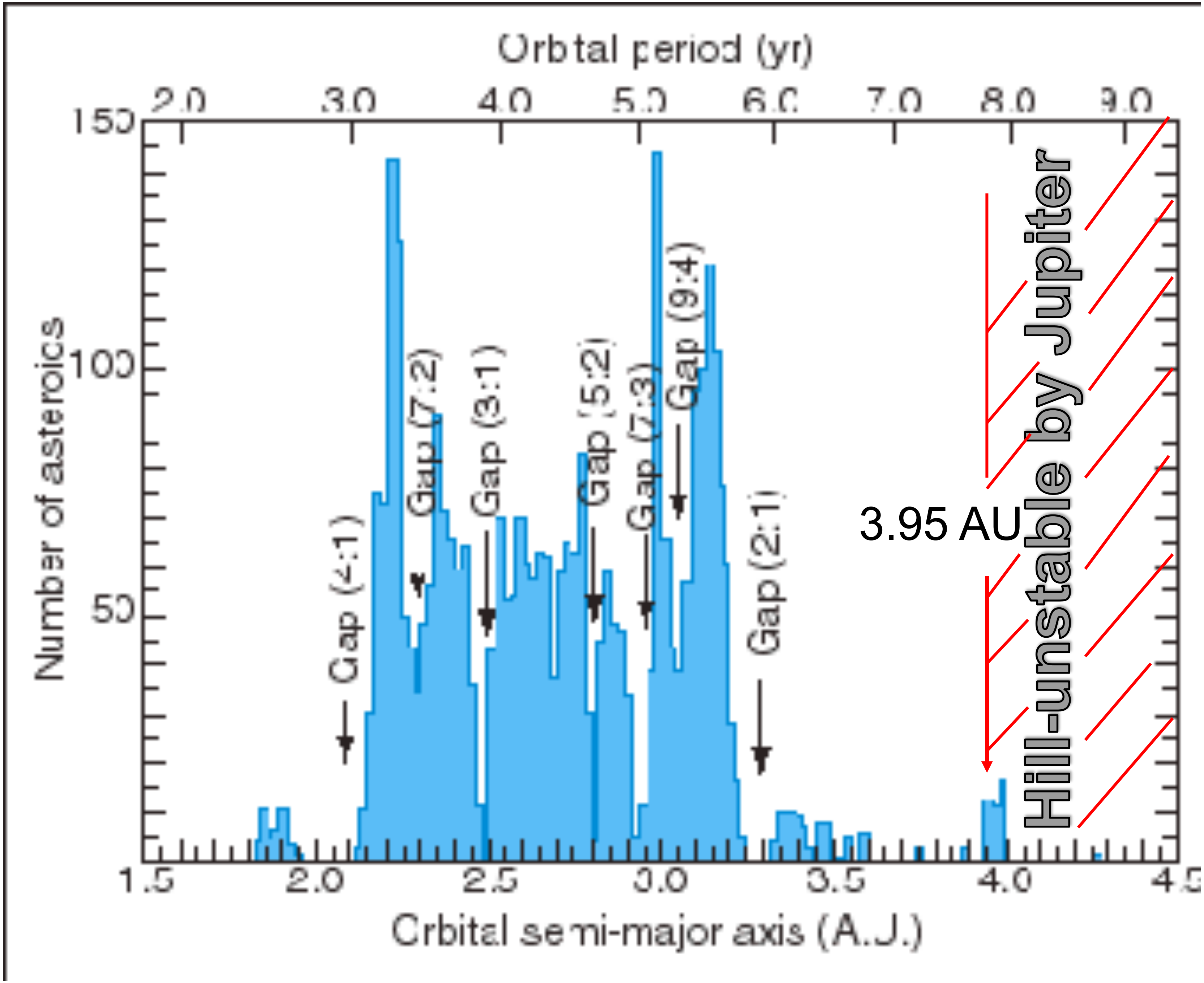
Since Jupiter is at $a_J = 5.2$ AU, the outermost Hill-stable circular orbit is at

$r = (1 - x)\ a_J = 0.76\ a_J = 3.95$ AU.

Asteroid belt objects are indeed found at $r <\sim 4$ AU.

Hildas and Thule group at ~4 AU are the outermost large groups of asteroids, except for the Trojan and Greek asteroids at Jupiter's $a = a_J = 5.2$ AU





Orbital period (yr)
2.0
3.0
4.0
5.0
6.0
7.0
8.0
9.0
150
100
50
Number of asteroids
Gap (4:1)
Gap (7:2)
Gap (3:1)
Gap (5:2)
Gap (7:3)
Gap (9:4)
Gap (2:1)
3.95 AU
Hill-unstable by Jupiter
1.5
2.0
2.5
3.0
3.5
4.0
4.5
Orbital semi-major axis (A.U.)