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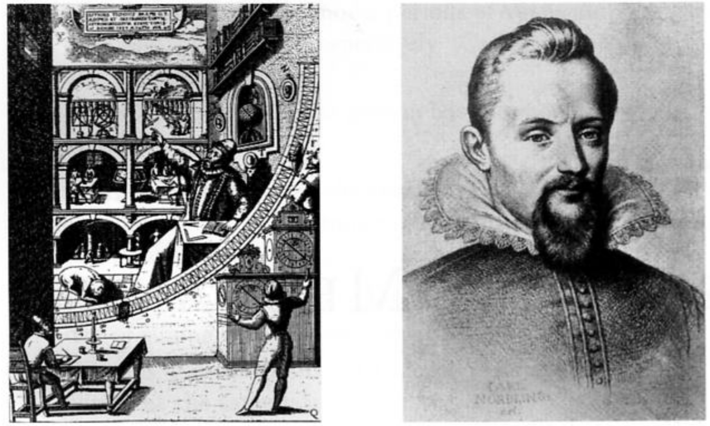
Chapter 2

CELESTIAL MECHANICS

2.1 Elliptical Orbits

Although the inherent simplicity of the Copernican model was aesthetically pleasing, the idea of a heliocentric universe was not immediately accepted; it lacked the support of observations capable of unambiguously demonstrating that a geocentric model was wrong. After the death of Copernicus, Tycho Brahe (1546–1601), the foremost naked-eye observer, carefully followed the motions of the "wandering stars" and other celestial objects. He carried out his work at the observatory, Uraniborg, on the island of Hveen (a facility provided for him by King Frederick II of Denmark). To improve the accuracy of his observations, Tycho used large measuring instruments, such as the quadrant depicted in the mural in Fig. 2.1(a). Tycho's observations were so meticulous that he was able to measure the position of an object in the heavens to an accuracy of better than $4'$, approximately one-eighth the angular diameter of a full moon. Through the accuracy of his observations he demonstrated for the first time that comets must be very distant, well beyond the Moon, rather than being some form of atmospheric phenomenon. Tycho is also credited with observing the supernova of 1572, which clearly demonstrated that the heavens were not unchanging as Church doctrine held. (This observation prompted King Frederick to build Uraniborg.) Despite the great care with which he carried out his work, Tycho was not able to find any clear evidence of the motion of Earth through the heavens, and he therefore concluded that the Copernican model must be false (see Section 3.1).

At Tycho's invitation, Johannes Kepler (1571–1630), a German mathematician, joined him at Uraniborg (Fig. 2.1b). Unlike Tycho, Kepler was a heliocentrist, and it was his desire to find a geometrical model of the universe



(a) (b)

Figure 2.1 (a) Mural of Tycho Brahe (1546–1601). (b) Johannes Kepler (1571–1630). (Courtesy of Yerkes Observatory.)

that would be consistent with the best observations then available, namely Tycho's. After Tycho's death, Kepler inherited the mass of observations accumulated over the years and began a painstaking analysis of the data. His initial, almost mystic, idea was that the universe is arranged with five perfect solids, nested to support the six known naked-eye planets (including Earth) on crystalline spheres, with the entire system centered on the Sun. After this model proved unsuccessful, he attempted to devise an accurate set of circular planetary orbits about the Sun, focusing specifically on Mars. Through his very clever use of offset circles and equants,[1] Kepler was able to obtain excellent agreement with Tycho's data for all but two of the points available. In particular, the discrepant points were each off by approximately $8'$, or twice the accuracy of Tycho's data. Believing that Tycho would not have made observational errors of this magnitude, Kepler felt forced to dismiss the idea of purely circular motion.

Rejecting the last fundamental assumption of the Ptolemaic model, Kepler began to consider the possibility that planetary orbits were elliptical in shape rather than circular. Through this relatively minor mathematical (though monumental philosophical) change, he was finally able to bring all of Tycho's observations into agreement with a model for planetary motion. This paradigm shift also allowed Kepler to discover that the orbital speed of a planet is not

[1]Recall the geocentric use of circles and equants by Ptolemy; see Fig. 1.3.

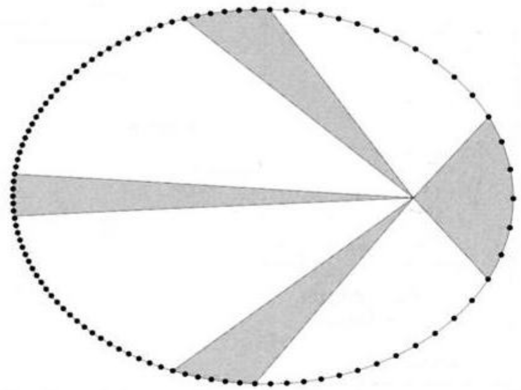


Figure 2.2 Kepler's second law states that the area swept out by a line between a planet and the focus of an ellipse is always the same for a given time interval, regardless of the planet's position in its orbit. The dots are evenly spaced in time.

constant but varies in a precise way depending on its location in its orbit. In 1609 Kepler published the first two of his three laws of planetary motion in the book, *Astronomica Nova*, or *The New Astronomy*:

> **Kepler's First Law** A planet orbits the Sun in an ellipse, with the Sun at one focus of the ellipse.

> **Kepler's Second Law** A line connecting a planet to the Sun sweeps out equal areas in equal time intervals.

Kepler's first and second laws are illustrated in Fig. 2.2, where each dot on the ellipse represents the position of the planet during evenly spaced time intervals.

Kepler's third law (also known as the harmonic law) was published ten years later in the book *Harmonica Mundi* (*The Harmony of the World*). His final law relates the average orbital distance of a planet from the Sun to its sidereal period:

> **Kepler's Third Law** $P^2 = a^3$,

where P is the orbital period of the planet, measured in *years*, and a is the average distance of the planet from the Sun, in *astronomical units*, or AU. An **astronomical unit** is, by definition, the average distance between Earth and the Sun, 1.496×10^{13} cm. A graph of Kepler's third law is shown in Fig. 2.3 using data for each planet in our solar system as given in Appendix B.

Important stuff! You must not only know these laws but understand how to use them (esp. 3rd law which will be discussed below).

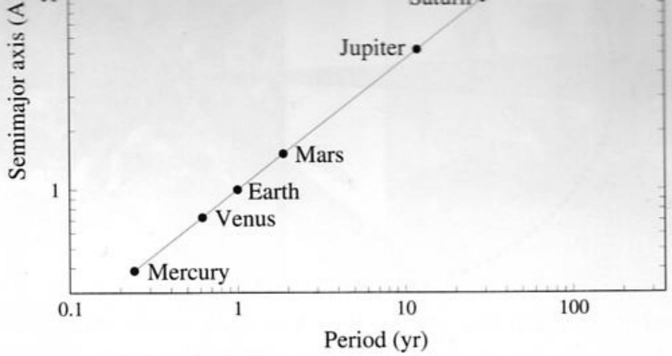


Figure 2.3 Kepler's third law for planets orbiting the Sun.

In retrospect it is easy to understand why the assumption of uniform and circular motion first proposed nearly 2000 years earlier was not found to be wrong much sooner; in most cases, planetary motion differs little from purely circular motion. In fact, it was actually fortuitous that Kepler chose to focus on Mars, since the data for that planet were particularly good and Mars deviates from circular motion more than most of the others.

To appreciate the significance of Kepler's laws, we must first understand the nature of the **ellipse**. An ellipse (see Fig. 2.4) is defined by that set of points that satisfies the equation

$$r + r' = 2a, \tag{2.1}$$

where a is a constant, known as the **semimajor axis** (half the length of the long axis of the ellipse) and r and r' represent the distances to the ellipse from the two **focal points**, F and F', respectively. Notice that if F and F' were located at the same point, then $r' = r$ and the previous equation would reduce to $r = r' = a$, the equation for a circle. Thus a circle is simply a special case of an ellipse. The distance b is known as the **semiminor axis**. The distance of either focal point from the center of the ellipse may be expressed as ae, where e is defined to be the **eccentricity** of the ellipse ($0 \le e < 1$). For a circle, $e = 0$.

A convenient relationship among a, b, and e may be found. Consider one of the two points at either end of the semiminor axis of an ellipse, where $r = r'$. In this case, $r = a$ and, by the Pythagorean theorem, $r^2 = b^2 + a^2e^2$. Substitution

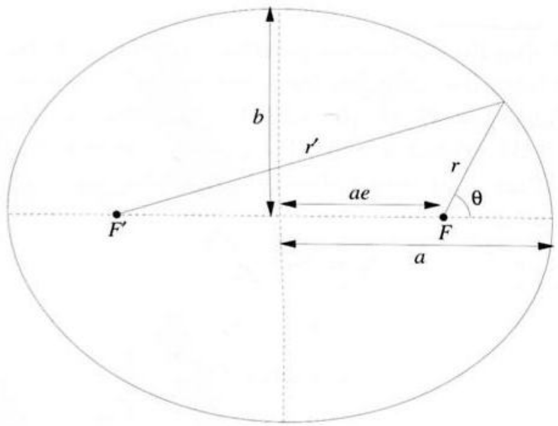


Figure 2.4 The geometry of an elliptical orbit.

leads immediately to the expression

$$b^2 = a^2(1 - e^2). \tag{2.2}$$

According to Kepler's first law, a planet orbits the Sun in an ellipse, with the Sun located at one focus of the ellipse, the **principal focus** (the other focus is empty space). The second law states that the orbital speed of a planet depends on its *location* in that orbit. To describe in detail the orbital behavior of a planet, it is necessary to specify where that planet is (its position vector) as well as how fast, and in what direction, the planet is moving (its velocity vector).

It is often most convenient to express a planet's orbit in polar coordinates, indicating its distance r from the principal focus in terms of an angle θ measured counterclockwise from the major axis of the ellipse (see Fig. 2.4). Using the Pythagorean theorem,

$$r'^2 = r^2 \sin^2\theta + (2ae + r\cos\theta)^2 ,$$

which reduces to

$$r'^2 = r^2 + 4ae(ae + r\cos\theta) .$$

Using the definition of an ellipse, $r + r' = 2a$, we find that

$$r = \frac{a\,(1 - e^2)}{1 + e\cos\theta} \qquad (0 \le e < 1). \tag{2.3}$$

It is left as an exercise to show that the total area of an ellipse is given by

$$A = \pi ab. \tag{2.4}$$

Eq. Of ellipse is very important, and how the apocenter and pericenter distances follow from it.

axis of Mars' orbit is 1.5237 AU (or 2.2794×10^{13} cm) and the planet's orbital eccentricity is 0.0934. When $\theta = 0°$, the planet is closest to the Sun, a point known as **perihelion**, and is at a distance given by

$$\begin{aligned} r_p &= \frac{a\,(1-e^2)}{1+e} \\ &= a\,(1-e) \\ &= 1.3814 \text{ AU.} \end{aligned} \tag{2.5}$$

Similarly, **aphelion** ($\theta = 180°$), the point where Mars is farthest from the Sun, is at a distance given by

$$\begin{aligned} r_a &= \frac{a\,(1-e^2)}{1-e} \\ &= a\,(1+e) \\ &= 1.6660 \text{ AU.} \end{aligned} \tag{2.6}$$

The variation in Mars' orbital distance from the Sun amounts to approximately 19% between perihelion and aphelion.

An ellipse is actually one of a class of curves known as conic sections, found by passing a plane through a cone (see Fig. 2.5). Each type of conic section has its own characteristic range of eccentricities. As already mentioned, a circle is a conic section having $e = 0$, and an ellipse has $0 \leq e < 1$. A curve having $e = 1$ is known as a **parabola** and is described by the equation

$$r = \frac{2p}{1+\cos\theta} \qquad (e = 1), \tag{2.7}$$

where p is the distance of closest approach to the parabola's *one* focus, at $\theta = 0$. Curves having eccentricities greater than unity, $e > 1$, are **hyperbolas** and have the form

$$r = \frac{a(e^2-1)}{1+e\cos\theta} \qquad (e > 1). \tag{2.8}$$

Each type of conic section is related to a specific form of celestial motion.

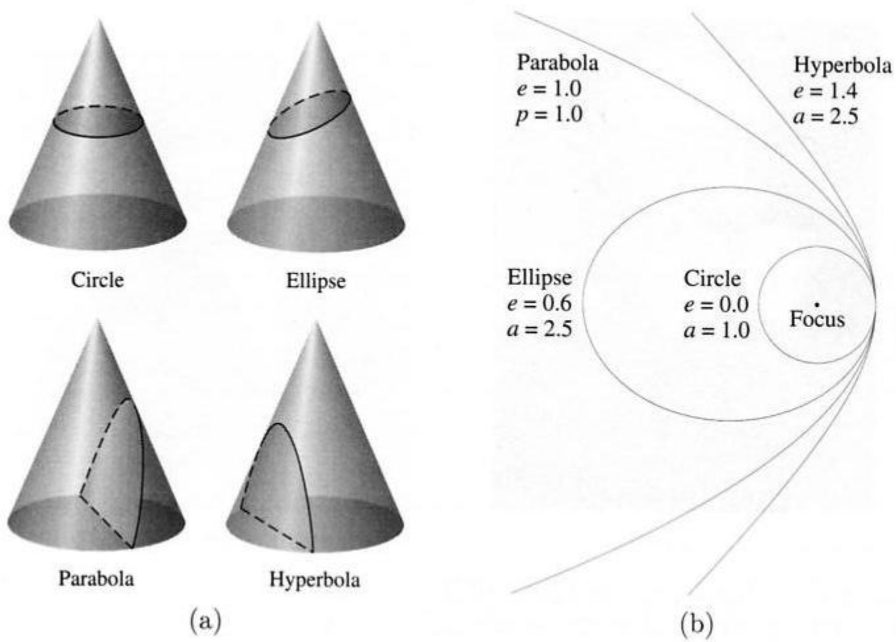


Figure 2.5 (a) Conic sections. (b) Related orbital motions.

2.2 Newtonian Mechanics

At the time Kepler was developing his three laws of planetary motion, Galileo Galilei (1564–1642), perhaps the first of the true experimental physicists, was studying the motion of objects on Earth (Fig. 2.6a). It was Galileo who proposed the earliest formulation of the concept of inertia; he had also developed an understanding of acceleration. In particular, he realized that objects near the surface of Earth fell with the same acceleration, independent of their weight. Whether Galileo publicly proved this fact by dropping objects of differing weights from the Leaning Tower of Pisa is a matter of some debate.

Galileo is also the father of modern observational astronomy. Shortly after learning about the 1608 invention of the first crude spyglass, he thought through its design and constructed his own. Using his new telescope to carefully observe the heavens, Galileo quickly made a number of important observations in support of the heliocentric model of the universe. In particular, he

(a) (b)

Figure 2.6 (a) Galileo Galilei (1564–1642). (b) Isaac Newton (1642–1727). (Courtesy of Yerkes Observatory.)

therefore was not a perfect sphere. Observations of the varying phases of Venus implied that the planet did not shine by its own power, but must be reflecting sunlight from varying positions in its orbit about the Sun. He also discovered that the Sun itself is blemished, possessing sunspots that varied in number and location. But perhaps the most damaging observation for the geocentric model, a model still strongly supported by the Church, was the discovery of four moons in orbit about Jupiter, indicating the existence of at least one other center of motion in the universe.

Many of Galileo's first observations were published in his book *Sidereus Nuncius* (*The Starry Messenger*) in 1610. By 1616 the Church forced him to withdraw his support of the Copernican model, although he was able to continue his study of astronomy for some years. In 1632 Galileo published another work, *The Dialogue on the Two Chief World Systems*, in which a three-character play was staged. In the play Salviati was the proponent of Galileo's views, Simplicio believed in the old Aristotelian view, and Sagredo acted as the neutral third party who was invariably swayed by Salviati's arguments. In a strong reaction, Galileo was called before the Roman Inquisition and his book was heavily censored. The book was then placed on the *Index* of banned books, a collection of titles that included works of Copernicus and Kepler. Galileo was put under house arrest for the remainder of his life, serving out his term at his home in Florence.

In 1992, after a 13-year study by Vatican experts, Pope John Paul II officially announced that, because of a "tragic mutual incomprehension," the Roman Catholic Church had erred in its condemnation of Galileo some 360 years earlier. By reevaluating its position, the Church demonstrated that, at least on this issue, there is room for the philosophical views of both science and religion.

In the year of Galileo's death, on Christmas day, Isaac Newton was born (1642–1727), arguably the greatest of all scientific minds (Fig. 2.6b). At age 18, Newton enrolled at Cambridge University and subsequently obtained his bachelor's degree. In the two years following the completion of his formal studies, and while living at home in Woolsthorpe, in rural England, away from the immediate dangers of the Plague, Newton engaged in what was likely the most productive period of scientific work ever carried out by one individual. During that interval, he made significant discoveries and theoretical advances in understanding motion, astronomy, optics, and mathematics. Although his work was not published immediately, the *Philosophiae Naturalis Principia Mathematica* (*Mathematical Principles of Natural Philosophy*), now simply known as the *Principia*, finally appeared in 1687 and contained much of his work on mechanics, gravitation, and the calculus. The publication of the *Principia* came about largely as a result of the urging of Edmond Halley, who paid for its printing. Another book, *Optiks*, appeared separately in 1704 and contained Newton's ideas about the nature of light and some of his early experiments in optics. Although many of his ideas concerning the particle nature of light were later shown to be in error (see Section 3.3), much of Newton's other work is still used extensively today.

Newton's great intellect is evidenced in his solution of the so-called brachistochrone problem posed by Johann Bernoulli, the Swiss mathematician, as a challenge to his colleagues. The brachistochrone problem amounts to finding the curve along which a bead could slide over a frictionless wire in the least amount of time while only under the influence of gravity. The deadline for finding a solution was set at a year and a half. The problem was presented to Newton late one afternoon; by the next morning he had found the answer by inventing a new area of mathematics known as the calculus of variations. Although the solution was published anonymously at Newton's request, Bernoulli commented, "By the claw, the lion is revealed."

Classical mechanics is described by Newton's three laws of motion, along with his universal *law of gravity.* Outside of the realms of atomic dimensions, velocities approaching the speed of light, or extreme gravitational forces, Newtonian physics has proven very successful in explaining the results of observations and experiments. Those regimes where Newtonian mechanics have been shown to be unsatisfactory will be discussed in later chapters.

Newton's first law of motion may be stated as:

> **Newton's First Law** The Law of Inertia. An object at rest will remain at rest and an object in motion will remain in motion in a straight line at a constant speed unless acted upon by an unbalanced force.

To establish whether an object is actually moving, a reference frame must be established. We will refer later to reference frames that have the special property that the first law is valid; all such frames are known as **inertial reference frames**. Noninertial reference frames are accelerated with respect to inertial frames.

The first law may be restated in terms of the momentum of an object, $\mathbf{p} = m\mathbf{v}$, where m and $\mathbf{v}$ are mass and velocity, respectively.[2] Thus Newton's first law may be expressed as *"the momentum of an object remains constant unless it experiences an unbalanced force."*[3]

The second law is actually a definition of the concept of force:

> **Newton's Second Law** The *net* force (the sum of all forces) acting on an object is proportional to the object's mass and its resultant acceleration.

If an object is experiencing n forces, then the net force is given by

$$\mathbf{F}_{\text{net}} = \sum_{i=1}^{n} \mathbf{F}_i = m\mathbf{a}. \tag{2.9}$$

However, since $\mathbf{a} = d\mathbf{v}/dt$, Newton's second law may also be expressed as

$$\mathbf{F}_{\text{net}} = m\frac{d\mathbf{v}}{dt} = \frac{d(m\mathbf{v})}{dt} = \frac{d\mathbf{p}}{dt}; \tag{2.10}$$

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[2] Hereafter, all vectors will be indicated by boldface type. Vectors are quantities described by both a magnitude and a direction.

[3] The law of inertia is an extension of the original concept developed by Galileo.

Figure 2.7 Newton's third law.

the net force on an object is equal to the time rate of change of its momentum, $\mathbf{p}$. $\mathbf{F}_{\text{net}} = d\mathbf{p}/dt$ represents the most general statement of the second law, allowing for a time variation in the mass of the object such as occurs with rocket propulsion.

The third law of motion is generally expressed as:

Newton's Third Law For every action there is an equal and opposite reaction.

In this law, *action* and *reaction* are to be interpreted as forces acting on different objects. Consider the force exerted *on* one object (object 1) *by* a second object (object 2), $\mathbf{F}_{12}$. Newton's third law states that the force on object 2 due to object 1, $\mathbf{F}_{21}$, must necessarily be of the same magnitude but in the opposite direction (see Fig. 2.7). Mathematically, the third law can be represented as

$$\mathbf{F}_{12} = -\mathbf{F}_{21}.$$

Using his three laws of motion along with Kepler's third law, Newton was able to find an expression describing the force that holds planets in their orbits. Consider the case of *circular* orbital motion of a mass m about a much larger mass M ($M \gg m$). Allowing for a system of units other than years and astronomical units, Kepler's third law may be written as

$$P^2 = kr^3,$$

where r is the distance between the two objects and k is a constant of proportionality. Thus the period of the orbit may be found from the constant velocity of m by

$$P = \frac{2\pi r}{v}.$$

Substituting into the previous equation gives

$$\frac{4\pi^2 r^2}{v^2} = kr^3.$$

$$\Omega = 2\pi / P$$

$$v_K = \Omega r$$

Remember that many equations involving P or v can be rewritten using the angular frequency = angular speed = Omega

Rearranging terms and multiplying both sides by m leads to the expression

$$m\frac{v^2}{r} = \frac{4\pi^2 m}{kr^2}.$$

The left-hand side of the equation may be recognized as the centripetal force for circular motion; thus

$$F = \frac{4\pi^2 m}{kr^2}$$

must be the gravitational force keeping m in its orbit about M. However, Newton's third law states that the magnitude of the force exerted on M by m must equal the magnitude of the force exerted on m by M. Therefore the form of the equation ought to be symmetric with respect to exchange of m and M. Expressing this symmetry explicitly and grouping the remaining constants into a new constant G, we arrive at the form of the *law of universal gravitation* found by Newton,

$$F = G\frac{Mm}{r^2}, \tag{2.11}$$

where $G = 6.67259\times10^{-8}$ dyne cm^2 g^{-2} (the *universal gravitational constant*).[4]

Newton's law of gravity applies to any two objects having mass. In particular, for an extended object (as opposed to a point mass), the force exerted by that object on another extended object may be found by integrating over each of their mass distributions.

Example 2.2 The force exerted by a *spherically symmetric* object of mass M on a point mass m may be found by integrating over rings centered along a line connecting the point mass to the center of the extended object (see Fig. 2.8). In this way all points on a specific ring are located at the same distance from m. Furthermore, due to the symmetry of the ring, the gravitational force vector associated with it is oriented along the ring's central axis. Once a general description of the force due to one ring is determined, it is possible to add up the individual contributions from all such rings throughout the entire volume of the mass M. The result will be the force on m due to M.

Let r be the distance between the centers of the two masses, M and m. R_0 is the radius of the large mass, and s is the distance from the point mass to a point on the ring. Due to the symmetry of the problem, only the component of the gravitational force vector along the line connecting the centers of the two objects needs to be calculated; the perpendicular components will cancel.

[4]Dyne is the measure of force in the cgs system of units: 1 dyne = 1 g cm s^{-2} = 10^{-5} newton.

Well, we call the l.h.s. centrifugal not centripetal force. In fact centripetal should have a minus sign…. It's all a matter of a frame of reference. We like the frames comoving with the planet, they prefer the inertial frame I guess.

The acceleration of the Moon caused by Earth's gravitational pull may also be calculated directly from

$$a_g = G\frac{M_\oplus}{r^2} = 0.27 \text{ cm s}^{-2},$$

in agreement with the value for the centripetal acceleration.

In astrophysics, as in any area of physics, it is often very helpful to have some understanding of the energetics of specific physical phenomena in order to determine whether these processes are important in certain systems. Some models may be ruled out immediately if they are incapable of producing the amount of energy observed. Energy arguments also often result in simpler solutions to particular problems. For example, in the evolution of a planetary atmosphere, the possibility of a particular component of the atmosphere escaping must be considered. Such a consideration is based on a calculation of the escape velocity of the gas particles.

The amount of energy (the work) necessary to raise an object of mass m a height h against a gravitational force is equal to the change in the *potential energy* of the system. Generally, the change in potential energy resulting from a change in position between two points is given by

$$U_f - U_i = \Delta U = -\int_{\mathbf{r}_i}^{\mathbf{r}_f} \mathbf{F} \cdot d\mathbf{r}, \tag{2.13}$$

where $\mathbf{F}$ is the force vector, $\mathbf{r}_i$ and $\mathbf{r}_f$ are the initial and final position vectors, respectively, and $d\mathbf{r}$ is the infinitesimal change in the position vector for some general coordinate system (see Fig. 2.9). If the gravitational force on m is due to a mass M located at the origin, then $\mathbf{F}$ is directed inward toward M, $d\mathbf{r}$ is directed outward, $\mathbf{F} \cdot d\mathbf{r} = -F dr$, and the change in potential energy becomes

$$\Delta U = \int_{r_i}^{r_f} G\frac{Mm}{r^2}\, dr.$$

Evaluating the integral, we have

$$U_f - U_i = -GMm\left(\frac{1}{r_f} - \frac{1}{r_i}\right).$$

Since only *relative changes* in potential energy are physically meaningful, a reference position where the potential energy is identically zero may be chosen. If, for a specific gravitational system, it is assumed that the potential energy

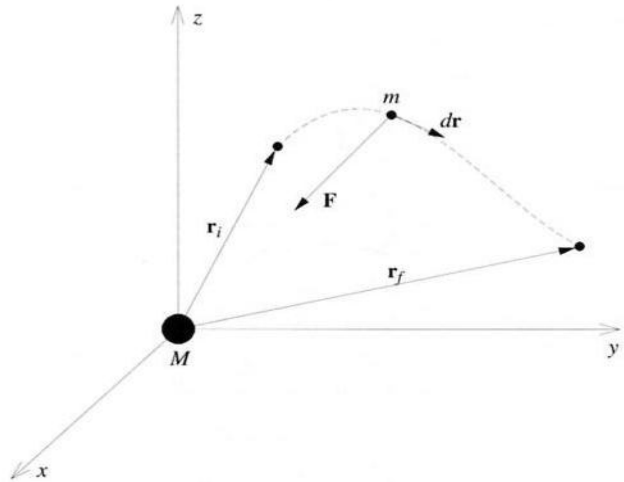

Figure 2.9 Gravitational potential energy. The amount of work done depends on the direction of motion relative to the direction of the force vector.

goes to zero at infinity, letting r_f approach infinity ($r_f \to \infty$) and dropping the subscripts for simplicity, gives

$$U = -G\frac{Mm}{r}. \tag{2.14}$$

Of course, the process may be reversed, the force may be found by *differentiating* the potential. For forces that depend only on r,

$$F = -\frac{\partial U}{\partial r}. \tag{2.15}$$

In a general three-dimensional description, $\mathbf{F} = -\nabla U$, where ∇U represents the *gradient* of U. In rectangular coordinates this becomes

$$\mathbf{F} = -\frac{\partial U}{\partial x}\hat{\mathbf{i}} - \frac{\partial U}{\partial y}\hat{\mathbf{j}} - \frac{\partial U}{\partial z}\hat{\mathbf{k}}.$$

Work must be performed on a massive object if its *speed*, $|\mathbf{v}|$, is to be changed. This can be seen by rewriting the work integral, first in terms of

$$
\begin{aligned}
W &\equiv -\Delta U \\
&= \int_{\mathbf{r}_i}^{\mathbf{r}_f} \mathbf{F} \cdot d\mathbf{r} \\
&= \int_{t_i}^{t_f} \frac{d\mathbf{p}}{dt} \cdot (\mathbf{v}\,dt) \\
&= \int_{t_i}^{t_f} m\frac{d\mathbf{v}}{dt} \cdot (\mathbf{v}\,dt) \\
&= \int_{t_i}^{t_f} m\left(\mathbf{v} \cdot \frac{d\mathbf{v}}{dt}\right) dt \\
&= \int_{t_i}^{t_f} m\frac{d\left(\frac{1}{2}v^2\right)}{dt}\, dt \\
&= \int_{v_i}^{v_f} m\, d\left(\frac{1}{2}v^2\right) \\
&= \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2.
\end{aligned}
$$

We may now identify the quantity

$$K = \frac{1}{2}mv^2 \tag{2.16}$$

as the *kinetic energy* of the object. Thus work done on the particle results in an equivalent change in the particle's kinetic energy. This statement is simply one example of the *conservation of energy*, a concept that is encountered frequently in all areas of physics.

Consider a particle of mass m, having an initial velocity $\mathbf{v}$ that is at a distance r from the center of a larger mass M, such as Earth. How fast must the mass be moving upward to escape completely the pull of gravity? To calculate the *escape velocity*, energy conservation may be used directly. The total initial mechanical energy of the particle (both kinetic and potential) is given by

$$E = \frac{1}{2}mv^2 - G\frac{Mm}{r}.$$

Assume that, in the critical case, the final velocity of the mass will be zero at a position infinitely far from M, implying that both the kinetic and potential

!
This is a general law, here used to derive escape velocity

energy will become zero. Clearly, by conservation of energy, the *total* energy of the particle must be identically zero at all times. Therefore

$$\frac{1}{2}mv^2 = G\frac{Mm}{r},$$

which may be solved immediately for the initial speed of m to give

$$v_{\rm esc} = \sqrt{2GM/r}. \tag{2.17}$$

Notice that the mass of the escaping object does not enter into the final expression for the escape velocity. Near the surface of Earth, $v_{\rm esc} = 11.2$ km s^{-1}.

2.3 Kepler's Laws Derived

Although Kepler did finally determine that the geometry of planetary motion was in the more general form of an ellipse rather than circular motion, he was unable to explain the nature of the force that kept the planets moving in their precise patterns. Not only was Newton successful in quantifying that force, he was also able to generalize Kepler's work, deriving the empirical laws of planetary motion from the gravitational force law. The derivation of Kepler's laws represented a crucial step in the development of modern astrophysics.

Although it is beyond the scope of this book, it can be shown that an elliptical orbit results from an attractive r^{-2} central force law such as gravity, when the total energy of the system is less than zero (a bound system). It can also be shown that a parabolic path is obtained when the energy is identically zero and that a hyperbolic path results from an unbounded system with an energy that is greater than zero. Newton was able to demonstrate the elliptical behavior of planetary motion and found that Kepler's first law must be generalized somewhat: The **center of mass** of the system, rather than the exact center of the Sun, is actually located at the focus of the ellipse. For our solar system, such a mistake is understandable, since the largest of the planets, Jupiter, has only 1/1000 the mass of the Sun. This places the center of mass of the Sun–Jupiter system near the surface of the Sun. Having used the naked-eye data of Tycho, Kepler can be forgiven for not realizing his error.

Before proceeding onward to derive Kepler's second and third laws, it will be useful to examine more closely the dynamics of orbital motion. An interacting two-body problem, such as binary orbits, or the more general many-body problem (often called the N-body problem), is most easily done in the reference frame of the center of mass.

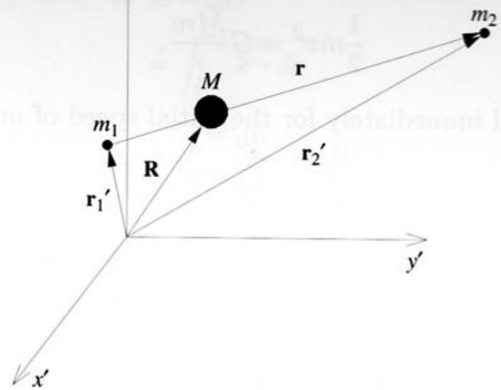

Figure 2.10 A general Cartesian coordinate system indicating the positions of m_1, m_2, and the center of mass (located at M).

Figure 2.10 shows two objects of masses m_1 and m_2 at positions $\mathbf{r}_1{}'$ and $\mathbf{r}_2{}'$, respectively, with the displacement vector from $\mathbf{r}_1{}'$ to $\mathbf{r}_2{}'$ given by

$$\mathbf{r} = \mathbf{r}_2{}' - \mathbf{r}_1{}'.$$

Define a position vector $\mathbf{R}$ to be a *weighted* average of the position vectors of the individual masses,

$$\mathbf{R} \equiv \frac{m_1\mathbf{r}_1{}' + m_2\mathbf{r}_2{}'}{m_1 + m_2}. \tag{2.18}$$

Of course, this definition could be immediately generalized to the case of n objects,

$$\mathbf{R} \equiv \frac{\sum_{i=1}^{n} m_i\mathbf{r}_i{}'}{\sum_{i=1}^{n} m_i}. \tag{2.19}$$

Rewriting the equation, we have

$$\sum_{i=1}^{n} m_i\mathbf{R} = \sum_{i=1}^{n} m_i\mathbf{r}_i{}'.$$

Then, if we define M to be the total mass of the system, $M \equiv \sum_{i=1}^{n} m_i$, the previous equation becomes

$$M\mathbf{R} = \sum_{i=1}^{n} m_i\mathbf{r}_i{}'.$$

Assuming that the individual masses do not change, differentiating both sides with respect to time gives

$$M\frac{d\mathbf{R}}{dt} = \sum_{i=1}^{n} m_i \frac{d\mathbf{r}_i{}'}{dt}$$

or

$$M\mathbf{V} = \sum_{i=1}^{n} m_i \mathbf{v}_i{}'.$$

The right-hand side is the sum of the linear momenta of every particle in the system, so the total linear momentum of the system may be treated as though *all* of the mass were located at $\mathbf{R}$, moving with a velocity $\mathbf{V}$. Thus $\mathbf{R}$ is the position of the center of mass of the system and $\mathbf{V}$ is the center-of-mass velocity. Letting $\mathbf{P} \equiv M\mathbf{V}$ be the linear momentum of the center of mass and $\mathbf{p}_i{}' \equiv m_i\mathbf{v}_i{}'$ be the linear momentum of an individual particle i, and again differentiating both sides with respect to time yields,

$$\frac{d\mathbf{P}}{dt} = \sum_{i=1}^{n} \frac{d\mathbf{p}_i{}'}{dt}.$$

If we assume that *all* of the forces acting on individual particles in the system are due to other particles contained within the system, Newton's third law requires that the total force must be zero. This constraint exists because of the equal magnitudes of action–reaction pairs. Of course, the momentum of individual masses may change. Using center-of-mass quantities, the total (or net) force on the system is

$$\mathbf{F} = \frac{d\mathbf{P}}{dt} = M\frac{d^2\mathbf{R}}{dt^2} = 0.$$

Therefore the center of mass will not accelerate if no external forces exist. This implies that a reference frame associated with the center of mass must be an inertial reference frame and the N-body problem may be simplified by choosing a coordinate system for which the center of mass is at rest at $\mathbf{R} = 0$.

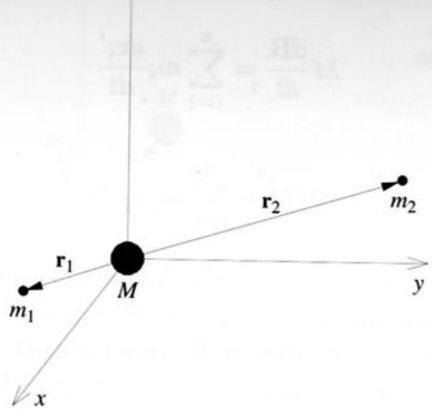

Figure 2.11 The center-of-mass reference frame for a binary orbit, with the center of mass fixed at the origin of the coordinate system.

If we choose a center-of-mass reference frame for a binary system, depicted in Fig. 2.11 ($\mathbf{R} = 0$), Eq. (2.18) becomes

$$\frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} = 0, \tag{2.20}$$

where the primes have been dropped, indicating center-of-mass coordinates. Both $\mathbf{r}_1$ and $\mathbf{r}_2$ may now be rewritten in terms of the displacement vector, $\mathbf{r}$. Substituting $\mathbf{r}_2 = \mathbf{r}_1 + \mathbf{r}$ gives

$$\mathbf{r}_1 = -\frac{m_2}{m_1 + m_2}\mathbf{r} \tag{2.21}$$

$$\mathbf{r}_2 = \frac{m_1}{m_1 + m_2}\mathbf{r}. \tag{2.22}$$

Next, define the *reduced mass* to be

$$\mu \equiv \frac{m_1 m_2}{m_1 + m_2}. \tag{2.23}$$

Then $\mathbf{r}_1$ and $\mathbf{r}_2$ become

$$\mathbf{r}_1 = -\frac{\mu}{m_1}\mathbf{r}$$

$$\mathbf{r}_2 = \frac{\mu}{m_2}\mathbf{r}.$$

Comment: the greek mu we use in the lectures on orbits most often denotes a slightly *different* quantity, (our mu)=m2/(m1+m2) = (their mu)/m1
While the reduced mass is a dimensional quantity, our mass parameter mu is a nondimensional mass of component 2 (by convention, a lighter component).

Notice that in stellar astrophysics mu also denotes mean molecular weight (nondimensional).

The convenience of the center-of-mass reference frame becomes evident when the total energy and orbital angular momentum of the system are considered. Including the necessary kinetic energy and gravitational potential energy terms, the total energy may be expressed as

$$E = \frac{1}{2}m_1 |\mathbf{v}_1|^2 + \frac{1}{2}m_2 |\mathbf{v}_2|^2 - G\frac{m_1 m_2}{|\mathbf{r}_2 - \mathbf{r}_1|}.$$

Substituting the relations for $\mathbf{r}_1$ and $\mathbf{r}_2$, along with the expression for the total mass of the system and the definition for the reduced mass, gives

$$E = \frac{1}{2}\mu v^2 - G\frac{M\mu}{r}, \tag{2.24}$$

where $v = |\mathbf{v}|$ and $\mathbf{v} \equiv d\mathbf{r}/dt$. We have also used the notation $r = |\mathbf{r}_2 - \mathbf{r}_1|$. The total energy of the system is equal to the kinetic energy of the reduced mass, plus the potential energy of the reduced mass moving about a mass M, assumed to be located at the origin. The distance between μ and M is equal to the separation between the objects of masses m_1 and m_2.

Similarly, the total orbital angular momentum,

$$\mathbf{L} = m_1\mathbf{r}_1 \times \mathbf{v}_1 + m_2\mathbf{r}_2 \times \mathbf{v}_2$$

becomes

$$\mathbf{L} = \mu\mathbf{r} \times \mathbf{v} = \mathbf{r} \times \mathbf{p}. \tag{2.25}$$

The total orbital angular momentum equals the angular momentum of the reduced mass only. *In general, the two-body problem may be treated as an equivalent one-body problem with the reduced mass moving about a fixed mass M at a distance r* (see Fig. 2.12).

To obtain Kepler's second law, it is necessary to consider the effect of gravitation on the orbital angular momentum of a planet. Using center-of-mass coordinates and evaluating the time derivative of the orbital angular momentum of the reduced mass (Eq. 2.25) gives

$$\begin{aligned}\frac{d\mathbf{L}}{dt} &= \frac{d\mathbf{r}}{dt} \times \mathbf{p} + \mathbf{r} \times \frac{d\mathbf{p}}{dt} \\ &= \mathbf{v} \times \mathbf{p} + \mathbf{r} \times \mathbf{F},\end{aligned}$$

the second expression arising from the definition of velocity and Newton's second law. Notice that, because $\mathbf{v}$ and $\mathbf{p}$ are in the same direction, their cross product is identically zero. Similarly, since $\mathbf{F}$ is a central force, directed inward

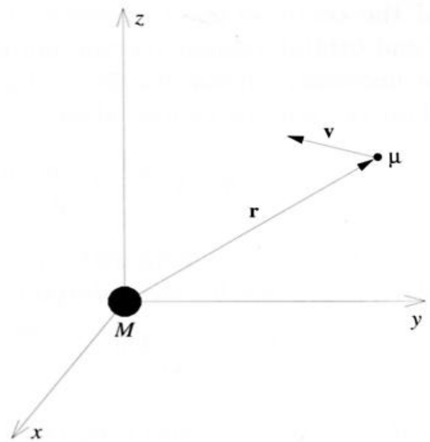

Figure 2.12 A binary orbit may be reduced to the equivalent problem of calculating the motion of the reduced mass, μ, about the total mass, M, located at the origin.

along $\mathbf{r}$, the cross product of $\mathbf{r}$ and $\mathbf{F}$ is also zero. The result is an important general statement concerning angular momentum:

$$\frac{d\mathbf{L}}{dt} = 0, \tag{2.26}$$

the angular momentum of a system is a constant for a central force law.

Since Kepler's second law relates the area of a section of an ellipse to a time interval, consider the infinitesimal area element in polar coordinates, as shown in Fig. 2.13:

$$dA = dr\,(r\,d\theta) = r\,dr\,d\theta.$$

If we integrate from the principal focus of the ellipse to a specific distance, r, the area swept out by an infinitesimal change in θ becomes

$$dA = \frac{1}{2}r^2\,d\theta.$$

Therefore the time rate of change in area swept out by a line joining a point on the ellipse to the focus becomes

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt}. \tag{2.27}$$

Now, the orbital velocity, $\mathbf{v}$, may be expressed in two components, one directed along $\mathbf{r}$ and the other perpendicular to $\mathbf{r}$. Letting $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ be the unit vectors

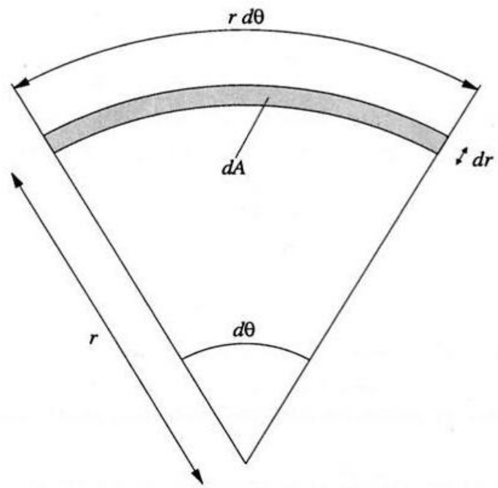

Figure 2.13 The infinitesimal area element in polar coordinates.

along **r** and its normal, respectively, **v** may be written as (see Fig. 2.14)

$$\begin{aligned}\mathbf{v} &= \mathbf{v}_r + \mathbf{v}_\theta \\ &= \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\theta}{dt}\hat{\theta}.\end{aligned} \tag{2.28}$$

Substituting v_θ into Eq. (2.27) gives

$$\frac{dA}{dt} = \frac{1}{2} r v_\theta.$$

Since **r** and $\mathbf{v}_\theta$ are perpendicular,

$$r v_\theta = |\mathbf{r} \times \mathbf{v}| = \left|\frac{\mathbf{L}}{\mu}\right| = \frac{L}{\mu}.$$

Finally, the time derivative of the area becomes

Kepler's Second Law (revisited)

$$\frac{dA}{dt} = \frac{1}{2}\frac{L}{\mu}. \tag{2.29}$$

It has already been shown that, because the orbital angular momentum is a constant, *the time rate of change of the area swept out by a line connecting a*

Here, you see the difference in our and OC's approach: we call the specific angular momentum and energy L and E, respectively. They'd call these quantities L/mu and E/mu

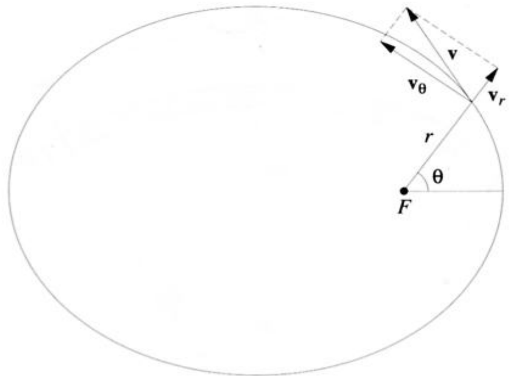

Figure 2.14 The velocity vector for elliptical motion in polar coordinates.

planet to the focus of an ellipse is a constant, one-half of the orbital angular momentum per unit mass. This is just Kepler's second law.

Simple expressions for both the total orbital angular momentum and the total energy may be found by computing their values at perihelion and aphelion and by invoking the appropriate conservation laws. Since at both perihelion and aphelion, **r** and **v** are perpendicular, the magnitude of the angular momentum simply becomes

$$L = \mu r v.$$

Recalling from Example 2.1 that $r_p = a(1-e)$ at perihelion and $r_a = a(1+e)$ at aphelion, and using the conservation of angular momentum,

$$\mu r_p v_p = \mu r_a v_a,$$

immediately reduces to

$$\frac{v_p}{v_a} = \frac{1+e}{1-e}.$$

Similarly, equating the total energy at both perihelion and aphelion provides a second expression relating v_p and v_a:

$$\frac{1}{2}\mu v_p^2 - G\frac{M\mu}{a(1-e)} = \frac{1}{2}\mu v_a^2 - G\frac{M\mu}{a(1+e)}.$$

Combining the two previous equations gives

$$v_p^2 = \frac{GM}{a}\left(\frac{1+e}{1-e}\right) \tag{2.30}$$

Again, we write L=r v_theta because our L is angular momentum per unit mass (their L/reduced mass). Since reduced mass in planetary problems is very close to the planet mass, we can think of our L as angular momentum per unit mass of a planet: m1=mass of the sun
m2=mass of a planet

$$m_1 >> m_2 \quad \Rightarrow$$

$$\mu = m_1 m_2 / (m_1 + m_2) \cong m_2$$

and

$$v_a^2 = \frac{GM}{a}\left(\frac{1-e}{1+e}\right). \tag{2.31}$$

Having found expressions for the orbital velocity at both perihelion and aphelion, the total orbital angular momentum may be easily calculated from

$$L = \mu r_p v_p,$$

which becomes

$$L = \mu\sqrt{GMa\left(1-e^2\right)}. \tag{2.32}$$

Note that L is a maximum for purely circular motion ($e = 0$) and goes to zero as the eccentricity approaches unity, as expected.

The total orbital energy may be found as well:

$$E = \frac{1}{2}\mu v_p^2 - G\frac{M\mu}{r_p}.$$

Making the appropriate substitutions, and after some rearrangement,

$$E = -G\frac{M\mu}{2a} = -G\frac{m_1 m_2}{2a}. \tag{2.33}$$

The total energy of a binary orbit depends only on the semimajor axis a and is exactly one-half the time-averaged potential energy of the system,

$$E = \frac{1}{2}\langle U \rangle,$$

where $\langle U \rangle$ denotes an average over one orbital period.[5] This is one example of the *virial theorem*, a general property of gravitationally bound systems. The virial theorem will be discussed in detail in Section 2.4.

A useful expression for the velocity of the reduced mass (or the relative velocity of m_1 and m_2) may be found directly by using the conservation of energy and equating the total orbital energy to the sum of the kinetic and potential energies:

$$-G\frac{M\mu}{2a} = \frac{1}{2}\mu v^2 - G\frac{M\mu}{r}.$$

Using the identity $M = m_1 + m_2$, this simplifies to give

$$v^2 = G\left(m_1 + m_2\right)\left(\frac{2}{r} - \frac{1}{a}\right). \tag{2.34}$$

[5] The proof that $\langle U \rangle = -GM\mu/a$ is left as an exercise.

Our E is energy per unit reduced mass (their E/reduced mass). Since reduced mass in planetary problems is very close to the planet mass, we can think of our E as total mechanical energy per unit mass of a planet.

This expression could also have been obtained directly by adding the vector components of orbital velocity. Calculating $\mathbf{v}_r$, $\mathbf{v}_\theta$, and v^2 will be left as exercises.

We are finally in a position to derive the last of Kepler's laws. Integrating the mathematical expression for Kepler's second law (Eq. 2.29) over one orbital period, P, gives the result

$$A = \frac{1}{2}\frac{L}{\mu}P.$$

Here the mass m orbiting about a much larger *fixed* mass M has been replaced by the more general reduced mass μ orbiting about the center of mass. Substituting the area of an ellipse, $A = \pi ab$, squaring the equation, and rearranging, produces the expression

$$P^2 = \frac{4\pi^2 a^2 b^2 \mu^2}{L^2}.$$

Finally, using Eq. (2.2) and the expression for the total orbital angular momentum (Eq. 2.32), the last equation simplifies to become

Kepler's Third Law (revisited)

$$P^2 = \frac{4\pi^2}{G\,(m_1 + m_2)}a^3. \tag{2.35}$$

! Pay attention,
the mass in this law
is the sum of masses

This is the general form of Kepler's third law. Not only did Newton demonstrate the relationship between the semimajor axis of an elliptical orbit and the orbital period, he also found a term not discovered empirically by Kepler, the square of the orbital period is inversely proportional to the total mass of the system. Once again Kepler can be forgiven for not noticing the effect. Tycho's data were for our solar system only and, because the Sun's mass $M_\odot$ is so much greater than the mass of any of the planets, $M_\odot + m_{\text{planet}} \simeq M_\odot$. Expressing P in years and a in astronomical units gives a value of unity for the collection of constants (including the Sun's mass).[6]

The importance to astronomy of Newton's form of Kepler's third law cannot be overstated. This law provides the most direct way of obtaining masses of celestial objects, a critical parameter in understanding a wide range of phenomena. Kepler's laws, as derived by Newton, apply equally well to planets orbiting the Sun, moons orbiting planets, stars in orbit about one another, and galaxy–galaxy orbits. Knowledge of the period of an orbit and the semimajor axis of the ellipse yields the total mass of the system. If relative distances to

[6] In 1621 Kepler was able to demonstrate that the four Galilean moons also obeyed his third law in the form $P^2 = ka^3$, where the constant k differed from unity. He did not attribute the fact that $k \neq 1$ to mass, however.

the center of mass are also known, the individual masses may be determined using Eq. (2.20).

Example 2.4 The orbital sidereal period of Io, one of the four Galilean moons of Jupiter, is 1.77 days = 1.53×10^5 s and the semimajor axis of its orbit is 4.22×10^{10} cm. Assuming that the mass of Io is insignificant compared to that of Jupiter, the mass of the planet may be estimated using Kepler's third law:

$$M_{\text{Jupiter}} = \frac{4\pi^2}{G}\frac{a^3}{P^2}$$
$$= 1.90 \times 10^{30} \text{ g}$$
$$= 318 \text{ M}_\oplus.$$

Appendix G contains a simple FORTRAN computer program, ORBIT, that makes use of many of the ideas discussed in this chapter. ORBIT will calculate, as a function of time, the location of a small mass that is orbiting about a much larger star (or it may be thought of as calculating the motion of the reduced mass about the total mass). Data generated by ORBIT were used to produce Fig. 2.2.

2.4 The Virial Theorem

In the last section we found that the total energy of the binary orbit was just one-half of the time-averaged gravitational potential energy (Eq. 2.33), or $E = \langle U \rangle /2$. Since the total energy of the system is negative, the system is necessarily bound. For gravitationally bound systems in equilibrium, it can be shown that the total energy is always one-half of the time-averaged potential energy; this is known as the **virial theorem**.

To prove the virial theorem, begin by considering the quantity

$$Q \equiv \sum_i \mathbf{p}_i \cdot \mathbf{r}_i,$$

where $\mathbf{p}_i$ and $\mathbf{r}_i$ are the linear momentum and position vectors for particle i in some inertial reference frame and the sum is taken to be over all particles in the system. The time derivative of Q is

$$\frac{dQ}{dt} = \sum_i \left(\frac{d\mathbf{p}_i}{dt} \cdot \mathbf{r}_i + \mathbf{p}_i \cdot \frac{d\mathbf{r}_i}{dt} \right).$$

You don't need to know the derivation of the virial theorem. You need to understand and remember what is what in it, or rather in many of it's forms.

Confused or not being able to recall factors of (-1/2) or (-1)? Here is an advice:

A good mnemonic (and not only mnemonic) is to know that the circular Keplerian orbit obeys the virial equation:
E = (-1/2)GM/a,
E_kin=(1/2)v_κ^2=(1/2)GM/a
E_pot=(-1)GM/a
E=E_kin + E_pot
In case of stars, E_kin would mean the kinetic energy of all its atoms, I.e. the total thermal energy, rather than the kinetic energy of the ordered motion of the planet.

Now, the left-hand side of the expression is just

$$\begin{aligned}\frac{dQ}{dt} &= \frac{d}{dt}\sum_i m_i \frac{d\mathbf{r}_i}{dt}\cdot\mathbf{r}_i \\ &= \frac{d}{dt}\sum_i \frac{1}{2}\frac{d}{dt}\left(m_i r_i^2\right) \\ &= \frac{1}{2}\frac{d^2 I}{dt^2},\end{aligned}$$

where

$$I = \sum_i m_i r_i^2$$

is the *moment of inertia* of the system of particles. So now

$$\frac{1}{2}\frac{d^2 I}{dt^2} - \sum_i \mathbf{p}_i \cdot \frac{d\mathbf{r}_i}{dt} = \sum_i \frac{d\mathbf{p}_i}{dt}\cdot\mathbf{r}_i. \tag{2.36}$$

The second term on the left-hand side is just

$$\begin{aligned}-\sum_i \mathbf{p}_i\cdot\frac{d\mathbf{r}_i}{dt} &= -\sum_i m_i \mathbf{v}_i\cdot\mathbf{v}_i \\ &= -2\sum_i \frac{1}{2} m_i v_i^2 \\ &= -2K,\end{aligned}$$

twice the negative of the total kinetic energy of the system. If we use Newton's second law, Eq. (2.36) becomes

$$\frac{1}{2}\frac{d^2 I}{dt^2} - 2K = \sum_i \mathbf{F}_i\cdot\mathbf{r}_i. \tag{2.37}$$

The right-hand side of this expression is known as the *virial of Clausius*, named after the physicist who first found this important energy relation.

If $\mathbf{F}_{ij}$ represents the force of interaction between two particles in the system (actually the force on i due to j), then, considering all of the possible forces acting on i,

$$\sum_i \mathbf{F}_i\cdot\mathbf{r}_i = \sum_i\Bigg(\sum_{\substack{j \\ j\neq i}} \mathbf{F}_{ij}\Bigg)\cdot\mathbf{r}_i.$$

If we use Newton's third law, $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$, so that

$$\sum_i \mathbf{F}_i \cdot \mathbf{r}_i = \frac{1}{2} \sum_i \left[\sum_{\substack{j \\ j \neq i}} (\mathbf{F}_{ij} - \mathbf{F}_{ji}) \right] \cdot \mathbf{r}_i. \tag{2.38}$$

After some manipulation, it can be shown that the virial of Clausius may be expressed as

$$\sum_i \mathbf{F}_i \cdot \mathbf{r}_i = \frac{1}{2} \sum_i \sum_{\substack{j \\ j \neq i}} \mathbf{F}_{ij} \cdot (\mathbf{r}_i - \mathbf{r}_j). \tag{2.39}$$

If it is assumed that the only contribution to the force is the result of the gravitational interaction between massive particles included in the system, then $\mathbf{F}_{ij}$ is

$$\mathbf{F}_{ij} = G \frac{m_i m_j}{r_{ij}^2} \hat{\mathbf{r}}_{ij},$$

where $r_{ij} = |\mathbf{r}_j - \mathbf{r}_i|$ is the separation between particles i and j and $\hat{\mathbf{r}}_{ij}$ is the unit vector directed from i to j:

$$\hat{\mathbf{r}}_{ij} \equiv \frac{\mathbf{r}_j - \mathbf{r}_i}{r_{ij}}.$$

Substituting the gravitational force into Eq. (2.39) gives

$$\begin{aligned} \sum_i \mathbf{F}_i \cdot \mathbf{r}_i &= -\frac{1}{2} \sum_i \sum_{\substack{j \\ j \neq i}} G \frac{m_i m_j}{r_{ij}^3} (\mathbf{r}_j - \mathbf{r}_i)^2 \\ &= -\frac{1}{2} \sum_i \sum_{\substack{j \\ j \neq i}} G \frac{m_i m_j}{r_{ij}}. \end{aligned} \tag{2.40}$$

The quantity

$$-G \frac{m_i m_j}{r_{ij}}$$

is just the potential energy U_{ij} between particles i and j. Note, however, that

$$-G \frac{m_j m_i}{r_{ji}}$$

also represents the same potential energy term and is included in the double sum as well, so the right-hand side of Eq. (2.40) includes the potential interaction between each pair of particles twice. Considering the factor of 1/2, Eq. (2.40) simply becomes

$$\sum_i \mathbf{F}_i \cdot \mathbf{r}_i = -\frac{1}{2} \sum_i \sum_{\substack{j \\ j \neq i}} G \frac{m_i m_j}{r_{ij}} = \frac{1}{2} \sum_i \sum_{\substack{j \\ j \neq i}} U_{ij} = U, \tag{2.41}$$

the total potential energy of the system of particles. Finally, substituting into Eq. (2.37) and taking the average with respect to time gives

$$\frac{1}{2}\left\langle \frac{d^2 I}{dt^2} \right\rangle - 2\langle K\rangle = \langle U\rangle . \tag{2.42}$$

The average of d^2I/dt^2 over some time interval τ is just

$$\left\langle \frac{d^2 I}{dt^2} \right\rangle = \frac{1}{\tau}\int_0^\tau \frac{d^2 I}{dt^2}\, dt \tag{2.43}$$

$$= \frac{1}{\tau}\left(\left.\frac{dI}{dt}\right|_\tau - \left.\frac{dI}{dt}\right|_0 \right). \tag{2.44}$$

If the system is periodic, as in the case for orbital motion, then

$$\left.\frac{dI}{dt}\right|_\tau = \left.\frac{dI}{dt}\right|_0$$

and the average over one period will be zero. Even if the system being considered is not strictly periodic, the average will still approach zero when evaluated over a sufficiently long period of time (i.e., $\tau \to \infty$), assuming of course that dI/dt is bounded. This would describe, for example, a system that has reached an equilibrium or steady-state configuration. In either case, we now have $\langle d^2I/dt^2\rangle = 0$, so

$$-2\langle K\rangle = \langle U\rangle . \tag{2.45}$$

This result is one form of the virial theorem. The theorem may also be expressed in terms of the total energy of the system by using the relation $\langle E\rangle = \langle K\rangle + \langle U\rangle$. Thus

$$\langle E\rangle = \frac{1}{2}\langle U\rangle , \tag{2.46}$$

just what we found for the binary orbit problem.

The virial theorem applies to a wide variety of systems, from an ideal gas to a cluster of galaxies. For instance, consider the case of a static star. In equilibrium a star must obey the virial theorem, implying that its total energy is negative, one-half of the total potential energy. Assuming that the star formed as a result of the gravitational collapse of a large cloud (a nebula), the potential energy of the system must have changed from an initial value of nearly zero to its negative static value. This implies that the star must have lost energy in the process, meaning that gravitational energy must have been radiated into space during the collapse. Applications of the virial theorem will be described in more detail in later chapters.

…see? Just like the relationship between the kinetc, potential, and total energy of a Keplerian planet

Suggested Readings

General

Kuhn, Thomas S., *The Structure of Scientific Revolutions*, Second Edition, Enlarged, University of Chicago Press, Chicago, 1970.

Westfall, Richard S., *Never at Rest: A Biography of Isaac Newton*, Cambridge University Press, Cambridge, 1980.

Technical

Arya, Atam P., *Introduction to Classical Mechanics*, Prentice Hall, Englewood Cliffs, NJ, 1990.

Clayton, Donald D., *Principles of Stellar Evolution and Nucleosynthesis*, McGraw-Hill, New York, 1968.

Fowles, Grant R., and Cassiday, George L., *Analytical Mechanics*, Fifth Edition, Harcourt Brace and Company, Fort Worth, 1993.

Marion, Jerry B., and Thornton, Stephen T., *Classical Dynamics of Particles and Systems*, Fourth Edition, Saunders College Publishing, Fort Worth, 1995.

Problems

2.1 Assume that a rectangular coordinate system has its origin at the center of an elliptical planetary orbit and that the coordinate system's x axis lies along the major axis of the ellipse. Show that the equation for the ellipse is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where a and b are the lengths of the semimajor axis and the semiminor axis, respectively.

2.2 Using the result of Problem 2.1, prove that the area of an ellipse is given by $A = \pi ab$.

2.3 (a) Beginning with Eq. (2.3) and Kepler's second law, derive general expressions for $\mathbf{v}_r$ and $\mathbf{v}_\theta$ for a mass m_1 in an elliptical orbit about a second mass m_2. Your final answers should be functions of P, e, a, and θ only.

(b) Using the expressions for $\mathbf{v}_r$ and $\mathbf{v}_\theta$ that you derived in part (a), verify Eq. (2.34) directly from $v^2 = v_r^2 + v_\theta^2$.

2.4 Derive Eq. (2.24) from the sum of the kinetic and potential energy terms for the masses, m_1 and m_2.

2.5 Derive Eq. (2.25) from the total angular momentum of the masses, m_1 and m_2.

2.6 By expanding Eq. (2.38) and rearranging, obtain Eq. (2.39).

2.7 (a) Assuming that the Sun interacts only with Jupiter, calculate the total orbital angular momentum of the Sun–Jupiter system. The semimajor axis of Jupiter's orbit is $a = 5.2$ AU, its orbital eccentricity is $e = 0.048$, and its orbital period is $P = 11.86$ yr.

(b) Estimate the contribution the Sun makes to the total orbital angular momentum of the Sun–Jupiter system. For simplicity, assume that the Sun's orbital eccentricity is $e = 0$, rather than $e = 0.048$. *Hint:* First find the distance of the center of the Sun from the center of mass.

(c) Making the approximation that the orbit of Jupiter is a perfect circle, estimate the contribution it makes to the total orbital angular momentum of the Sun–Jupiter system. Compare your answer with the difference between the two values found in parts (a) and (b).

(d) Recall that the moment of inertia of a solid sphere of mass m and radius r is given by $I = \frac{2}{5}mr^2$, and that when the sphere spins on an axis passing through its center, its rotational angular momentum may be written as

$$L = I\omega,$$

where ω is the angular frequency measured in rad s^{-1}. Assuming (incorrectly) that both the Sun and Jupiter rotate as solid spheres, calculate approximate values for the rotational angular momenta of the Sun and Jupiter. Take the rotation periods of the Sun and Jupiter to be 26 days and 10 hours, respectively. The radius of the Sun is 6.96×10^{10} cm, and the radius of Jupiter is 6.9×10^{9} cm.

(e) What part of the Sun–Jupiter system makes the largest contribution to the total angular momentum?

2.8 (a) Using data contained in Problem 2.7 and in the chapter, calculate the escape velocity at the surface of Jupiter.

(b) Calculate the escape velocity from the solar system, starting from Earth's orbit. Assume that the Sun constitutes all of the mass of the solar system.

2.9 (a) The Hubble Space Telescope is in a nearly circular orbit, approximately 380 miles above the surface of Earth. Estimate its orbital period.

(b) Communications and weather satellites are often placed in *geosynchronous* "parking" orbits above Earth. These are orbits where satellites can remain fixed above a specific point on the surface of Earth. At what altitude must these satellites be located?

(c) Is it possible for a satellite in a geosynchronous orbit to remain "parked" over any location on the surface of Earth? Why or why not?

2.10 In general, an *integral average* of some continuous function $f(t)$ over an interval τ is given by

$$\langle f(t)\rangle = \frac{1}{\tau}\int_0^{\tau} f(t)\,dt.$$

Beginning with an expression for the integral average, prove that

$$\langle U\rangle = -G\frac{M\mu}{a},$$

a binary system's gravitational potential energy, averaged over one period, equals the value of the instantaneous potential energy of the system when the two masses are separated by the distance a, the semimajor axis of the orbit of the reduced mass about the center of mass. *Hint:* You may find the following definite integral useful;

$$\int_0^{2\pi}\frac{d\theta}{1+e\cos\theta} = \frac{2\pi}{\sqrt{1-e^2}}.$$

2.11 Cometary orbits usually have very large eccentricities, often approaching (or even exceeding) unity. Halley's comet has an orbital period of 76 yr and an orbital eccentricity of $e = 0.9673$.

(a) What is the semimajor axis of Comet Halley's orbit?

(b) Use the orbital data of Comet Halley to estimate the mass of the Sun.

(c) Calculate the distance of Comet Halley from the Sun at perihelion and aphelion.

(d) Determine the orbital speed of the comet when at perihelion, at aphelion, and on the semiminor axis of its orbit.

(e) How many times larger is the kinetic energy of Halley's comet at perihelion when compared to aphelion?

2.12 **Computer Problem** Using ORBIT, the FORTRAN computer code found in Appendix G, together with the data given in Problem 2.11, estimate the amount of time required for Halley's comet to move from perihelion to a distance of 1 AU away from the principal focus.

2.13 **Computer Problem** The computer code ORBIT (Appendix G) can be used to generate orbital positions, given the mass of the central star, the semimajor axis of the orbit, and the orbital eccentricity. Using ORBIT to generate the data, plot on a single sheet of graph paper the orbits for three hypothetical objects orbiting our Sun. Assume that the semimajor axis of each orbit is 1 AU and that the orbital eccentricities are:

(a) 0.0.

(b) 0.4.

(c) 0.9.

Note: Indicate the principal focus, located at $x = 0.0, y = 0.0$.

2.14 **Computer Problem**

(a) From the data given in Example 2.1, use ORBIT (Appendix G) to generate an orbit for Mars. Plot at least 25 points, evenly spaced in time, on a sheet of graph paper and clearly indicate the principal focus.

(b) Using a compass, draw a perfect circle on top of the elliptical orbit for Mars, choosing the radius of the circle and its center carefully in order to make the best possible approximation of the orbit. Be sure to mark the center of the circle you chose.

(c) What can you conclude about the merit of Kepler's first attempts to use offset circles and equants to model the orbit of Mars?

2.15 Given that a geocentric universe is (mathematically) only a matter of the choice of a reference frame, explain why the Ptolemaic model of the universe was able to survive scrutiny for such a long period of time.