ASTC25 (PLANETARY SYSTEMS) - PREPARATION FOR FINAL EXAM (COVERS THE COURSE SUBJECTS AFTER MIDTERM). WITH SOME SOLUTIONS/HINTS.

This set will be improved a bit a time goes on. It is already faily usable. If some data are missing, or formulation needs to be improved or made more explicit, this is an exercise for you. Some problems for which no detailed solutions were provided are good ideas for a problem that you can improve upon.

Some of the tasks require more time than those you'll have in the actual final exam. Nevertheless, they're all a great training for the exam.
Sometimes we use engineering notation (for example, 1.2e-3 for $1.2\cdot 10^{-3}$). Also, notice that some problems may have been already given as assignments or discussed partly in tutorials. Use the methododology of solving astro problems: 1. make a sketh if appropriate, 2. explain concepts and analytical manipulations of symbols; 3. units checks, 4. numerical evaluation of answers; 5. final check: are the results plausible?

1 [4p] Triangular points

Prove that independent of mass ratio $\mu = m_2/(m_1+m_2)$, the L4 and L5 Lagrange points in cR3B (restricted circular 3-body problem) are forming an equilateral triangle with masses m_1 and m_2. Do the calcultions in noninertial system corotating with the binary, centered on the center of mass; show that L4,5 are equilibrium points.

SOLUTION

This problem requires a careful placement of bodies: mass 1 has mass $(1-\mu)M$ where $M = m_1+m_2$ is the total mass and is located at $R_1 = (x,y) = (-\mu a, 0)$, while mass 2 has mass μM and is located at $R_2 = (x,y) = ((1-\mu)a, 0)$. The triangular point is located half-way along x-axis, and the height of the triangle is equal $\sqrt{3}a/2$. Points 4 and 5 are symmetrically located so it sufficed to prove the vanishing of net acceleration at L4 located above x-axis at $\mathbf{r} = (x,y) = (a(-\mu+1/2), \sqrt{3}a/2$. Let's write the gravitational accelerations, followed by the centrifugal acceleration (Coriolis acceleration is zero because the test particle is at rest in rotating system):

$$\mathbf{f} = -\frac{(1-\mu)GM\mathbf{r_1}}{r_1^3} - \frac{\mu GM\mathbf{r_2}}{r_2^3} + \Omega^2\mathbf{r},$$

where $\Omega^2 = GM/a^3$, $\mathbf{r_i} = \mathbf{r} - \mathbf{R_i}$ $(i = 1,2)$.

We have $\mathbf{r_1} = (a/2, \sqrt{3}a/2)$ and $\mathbf{r_2} = (-a/2, \sqrt{3}a/2)$ which, expectedly, give the lengths of these vectors as $r_1 = r_2 = a$ (sides of *equilateral* triangle). This simplifies the net acceleration to the form

$$\mathbf{f} = \Omega^2\left[-(1-\mu)\mathbf{r_1} - \mu\mathbf{r_2} + \mathbf{r}\right]$$

Writing out (x,y)-components of the square bracket:

$$-(1-\mu)(a/2, \sqrt{3}a/2) - \mu(-a/2, \sqrt{3}a/2) + (a/2-\mu a, \sqrt{3}a/2)$$

$$= (-a/2+\mu a+\mu a/2+a/2-\mu a,\ -\sqrt{3}a/2+\sqrt{3}a/2) = (0,0)$$

as required. This proves that L4 and L5 are equilibrium points.

Stability would be a more complicated calculation, but you do remember the result stated in one of the lecture notes? There is a critical mass ratio below which the triangular points are stable to small perturbations. Find out from the lectures, this could be good to know for the quiz.

2 [6p] Instability of collinear Lagrange points in Hill's approximation

Hill's equations approximate the dynamics around a small-mass planet in Cartesian coordinates (x,y) in which x is the radial axis on which two massive points of the circular, planar R3B are located, and y is the axis pointing in the direction in which the small secondary body moves in inertial frame (azimuthal direction). If all distances are nondimensional ratios of actual distances to the Roche lobe of the small planet, $r_L = (\mu/3)^{1/3}a$, and the time is non-dimensionalized as well by division through dynamical time Ω^{-1}, then the Hills equations take the form

$$\ddot{x} = -3x/r^3 + 3x + 2\dot{y}$$

$$\ddot{y} = -3y/r^3 - 2\dot{x}$$

Show that small-scale motion around either Lagrance point at (±1,0) is unstable, find the timescale of instability.

SOLUTION

Knowing that the Hills equations are symmetric w.r.t. planet and do not distinguish the vicinity of the L1 and L2 points, we can show their instability by restricting attention to the outer point at $(1,0)$, where we write $(x,y) = (1+\xi,\zeta)$. Variables $|\xi|,|\zeta| \ll 1$ are small deviations in x and y directions, whose squares and higher powers are negligible compared with the constants and their first power.

To that order of accuracy, the distance r to the planet satisfies $r^2 = x^2 + y^2 \simeq (1+\xi)^2 + \zeta^2 = 1 + 2\xi$, and $r^{-3} \simeq (1+2\xi)^{-3/2} \simeq 1 - 3\xi$. Keeping only constant and linear terms, equations of motion simplify to

$$\ddot{\xi} = (-3-3\xi)(1-3\xi) + (3+3\xi) + 2\dot{\zeta} \simeq 9\xi + 2\dot{\zeta}$$

$$\ddot{\zeta} = -3\zeta - 2\dot{\xi}$$

So far our equations say that for $\xi = \dot{\xi} = 0$ and $\zeta = \dot{\zeta} = 0$ the accellerations vanish, i.e. the L point is indeed an equilibrium point. So far so good.

To probe the stability, it is a standard procedure to try whether the exponential solitions $\xi = pe^{i\omega t}$, and $\zeta = qe^{i\omega t}$, where $(p,q = const)$, satisfy the linearized equations. Upon substitution,

$$0 = (9+\omega^2)\xi + 2i\omega\zeta$$

$$0 = -2i\omega\xi + (\omega^2 - 3)\zeta$$

Substitute ξ from the 2nd equation into the 1st,

$$(9-\omega^2)[(\omega^2-3)/(2i\omega)]\zeta + 2i\omega\zeta = 0$$

This equation is has non-trivial solutions ($\zeta \neq 0$) if

$$(9-\omega^2)(\omega^2-3) - 4\omega^2 = 0$$

or

$$\omega^2 = 1 \pm 2\sqrt{7}.$$

One of the solutions for ω^2 is negative and results in

$$\omega = \pm i[2\sqrt{7} - 1]^{1/2}.$$

The negative root $\omega = -\sqrt{2\sqrt{7}-1}$ signifies instability because exp $i\omega t = \exp[\sqrt{2\sqrt{7}-1}\,t]$ describes an exponential increase of a small separation from the collinear L-point. The instability occurs for all mass ratios (the Hill's equations can only demonstrate instability for $\mu \ll 1$, an underlying assumption in Hill's approximation).

3 3p] Maximum eccentricity

A system of two planets with equal masses exchanges angular mementum but not energy between the planets, so that the semi-major axes $a_1 = 1$ AU and $a_2 = 0.64$ AU remain constant. The first planet has initial eccentricity $e_{10} = 0.3$ and the second has $e_{10} = 0.1$. What is the maximum eccentricity that can be achieved by planet 2 in interaction with planet 1?

Hint: The sum (or total) of angular momenta of the planets is constant; write it out in the initial state and consider possible changes of e_1, which result in increase of e_2.

Ans: 0.348

4 [2p] Prove the gravitational focusing factor formula

$$b/R = \sqrt{1 + \frac{v_{esc}^2}{v^2}}$$

HINT: See the outline on a slide in lecture notes, write it down.

5 [2p] What is the minimum orbital period of motion of a silicate rock with $A = 0.4$ around the sun?

Reminder: all silicates evaporate at 2000 K, some already at 1600 K.

SOLUTION:

First, forget about the high temperature issue. Absolute minimum of period is achieved on a circular orbit with a = R (radius equal to the radius of the sun, which is R = 696000 km). From Kepler's law applied twice, once to the calculated orbit and once to Earth (which has period P = 365.25 days w.r.t. stars), we obain by side-by-side division

$$P_{min} = 365.25\text{days}\ (R/1AU)^{3/2} = 365.25(0.696/149.6)^{3/2}\text{days},$$

or in other words 0.1159 days, or 2.782 hours.

But that's likely incorrect since a silicate rock, absorbing most of incoming radiation, will evaporate near the surface of our star. Let's take maximum temperature of a silicate to be T = 1800 K. (In exam, you can assume a any particular value that youi can substantiate.) We know a formula for equilibrium temperature of a spherical rock near the sun:

$$T = 280K(1-A)^{1/4}(1AU/r)^{1/2} \, .$$

It applies to bodies with Bond albedo A. Equating T with 1800 K, A with 0.4, and squaring both sides, we obtain

$$(1800/280)^2 = 0.6^{1/2}(1AU/r) \, .$$

or

$$r/\mathrm{AU} = (28/180)^2 \, \sqrt{0.6} \, .$$

which can be plugged into the Kepler law to yield

$$P_{min}(T) = 365.25 \, (28/180)^3 \, 0.6^{3/4} \text{ days}$$

or in other words 0.937 days (22.5 hours).

6 [1p] Distance to the outermost planet

In a faraway planetary system, three planets are in 1:2:5 mean motion resonance. If the nearest one is at 0.1 AU from the star, what is the mean distance from the star to the outermost planet? Can we compute the stellar mass from the above data?

Suppose that the stellar mass is leaving the system radially, without affecting angular momenta of the planets, or their sum. After the mass loss removes 25 percent of stellar mass, how far will be the outermost planet?

7 [2p] Colonization of the Galaxy

Consider somepossible ways for humankind to travel throughout the Galaxy in order to colonize its habitable planets, including (i) using gravitational flybys of outer planets like Voyager spacecraft (find the current speed of Voyagers somewhere), (ii) radiation pressure sails (propose the material for the cosmic sail, its density and albedo, find the thickness of a 100m x 100m sail that guarantees the escape from the sun. Assume attached spacecraft mass equal 1000 kg. Propose the best starting location for the cosmic sailboat.

Estimate the times to reach the Galactic Center.

HINT: The force of gravity must exceed the force of radiation pressure, which is the rate of interception of photons' momentum times two (why times tow? explain connection to albedo).

8 [3p] NEA

A near-Earth asteroid (NEA) occupies a circular orbit that threatens a collision with Earth in the next century (don't be afraid, the problem is made-up). The mass of the asteroid is 1e15 kg. To avoid collision, it is enough to change its orbital radius from $a = 1.01$ to 1.02.

A couple of methods have been proposed. Which have a chance of deflecting the asteroid and which do not?

1. Send a kinetic projectile of mass 100 tonnes, i.e. 1e5 kg, to hit and push the asteroid off its track.

SOLUTION

Pushing across the direction of motion will not change the energy or semi-major axis. We will need to push along the orbit for most effective energy and ang. mom. transfer. Angular momentum change or energy change can be computed, from which the change of a (denoted as Δa) can be found and compared with the required one.

2. Plant and explode a thermonuclear bomb on its surface. The mass of hydrogen is 10 kg and the explosion releases 0.5 percent of the rest energy of hydrogen. Assume ten percent of energy is transferred to the orbital energy.

SOLUTION

Again, via ΔE, while $da/a = dE/E$ and hence $(\Delta a)/a = (\Delta E)/E$.

3. Place a massive object (1000 tonnes) at a distance of 20 km in front of the asteroid, in its precise path, such that the gravitational tug exerted over a period of 100 years tows the NEA into a higher heliocentric orbit.

SOLUTION

The force between the NEA and a 'tug boat' is GMm/r^2, and the NEA forward acceleration is Gm/r^2, where M is the asteroid mass and m the object's mass, while $r = 20$ km.

The specific angular momentum of NEA is $L = \sqrt{GM_\odot a}$ and its rate of change, specific torque, is the acceleration times radius: $dL/dt = Gma/r^2$. On the other hand, differentiation of L leads to

$$\frac{dL}{dt} = \frac{1}{2}\sqrt{\frac{GM_\odot}{a}}\frac{da}{dt}.$$

This gives

$$\frac{da}{dt} = 2G^{1/2}m\frac{a^{3/2}}{r^2\sqrt{M_\odot}}$$

Over time $\Delta t = 100$ yr, the change of semi-major axis a thus equals $\Delta t(da/dt)$, or

$$\Delta a = 2G^{1/2}m\frac{a^{3/2}}{r^2\sqrt{M_\odot}}\Delta t.$$

Units check out ok (you always are required to verify them, even if we don't always do it here).

Numerically, $\Delta a = ...$ Check whether the method works!

9 [5p] Temperature of a large solid particle

What is the dependence of the equilibrium particle temperature on the distance from a star of luminosity $L = 8L_\odot$, if the particle scatters $A = 50\%$ of the visible radiation, and radiates the infrared thermal flux with wavelength-independent efficiency $Q_{abs} = 1$? (Kirchhoffs law states that absorption and emission efficiencies are equal, hence "abs"). Express your result in the form $T(r) = const\,(r/AU)^{const}$ and find T at $r = 80$ AU.

Hint: Albedo $A = 0.5$ is that effective part of the cross section area of a body which scatters starlight. Q_{abs} is that effective part of the total area of a body which radiates thermalradiation into space. Both coefficients are dimensionless.

SOLUTION

The percentage of starlight scattered is called albedo, A. In this case $A = 0.5$, but we keep it as A to obtain a more general answer. Thus, $(1-A)L/(4\pi r^2)$ is the flux of energy absorbed by a particle of cross-sectional area πs^2, where s is the radius of the particle, located at distance r from the star. Energy absorbed per unit time is thus $dE/dt = (1-A)Ls^2/(4r^2)$, must on the other hand be totally radiated away in the thermal (infrared radiation wavelength range. If T is the surface temperature of the particle and is constant on its whole surface are $4\pi s^2$, then the flux from the particle equals $Q_{abs,IR}\sigma T^4 4\pi s^2$. Equality of absorbed and re-radiated energy gives:

$$T^4 = \frac{1-A}{Q_{abs,IR}} \frac{L}{16\pi\sigma r^2}$$

.

Hence, $T \sim (r/\mathrm{AU})^{-1/2}$, and the front coefficient is different from the case of the sun anda blackbody (perfect absorpion, perfect emission), where it would be equal to about 280 K, by a factor $[(1-A)/Q_{abs,IR}(L/L_\odot)]^{1/4}$. If $L = 8L_\odot$, $A = 0.5$, and $Q_{abs,IR} = 1$, then $T \simeq 396(r/\mathrm{AU})^{-1/2}$ K, which gives $T(r = 80$ AU) = 44 K. This, and lower temperatures are typical for the Trans-Neptunian Objects. This is also approximately the temperature of the coldest known satellite in our system, Triton (satellite of Neptune).

10 [6p] Temperature of a small dust grain

What is the dependence of temperature of a small dust grain on the distance from the star (of luminosity $8L_\odot$) if the particle absorbs 50% of the visible radiation, and radiates the infrared ($\lambda > 10\mu$m) thermal flux with a wavelength-dependent efficiency given by the formula

$$Q_{abs,IR} = \lambda_0/\lambda$$

where $\lambda_0 = 10\mu$m?

For simplicity, substitute for the emitted λ in this formula an effective wavelength λ_{eff} provided by the Wien's law of black body radiation (a λ at which Planck curve with temperature T peaks). Wilhelm Wien got the Nobel prize in 1911 for thus formula:

$$\lambda_{eff} = \frac{2900\mu\mathrm{m}\cdot\mathrm{K}}{T}.$$

Express your result in the form $T(r) = const(r/AU)^{const}$ and find $T(r = 80$ AU).

SOLUTION

Although this problem can be solved step by step, just like the previous one, we can use the previous solution to simplify the derivation a little. The power absorbed by the spherical particle of radius s at a distance r from the star is, again, $dE/dt = (1-A)Ls^2/(4r^2)$.

The flux emitted from the particle in infrared is, again, $Q_{abs,IR}\sigma T^4 4\pi s^2$, but now is emitted at an effective wavelength λ_{eff} with efficiency $Q_{abs,IR}(\lambda_{eff}) = \lambda_0/\lambda_{eff}$, where $\lambda_{eff} = \frac{2900\mu\mathrm{m}\cdot\mathrm{K}}{T}$.

Equating the absorbed and re-radiated energy gives:

$$T^4 = \frac{1-A}{Q_{abs,IR}(\lambda_{eff})} \frac{L}{16\pi\sigma r^2}$$

which divided by the result of the previous problem (let's call that other temperature T_1) gives

$$T^4/T_1^4 = (290\mathrm{K}/T)$$

or, considering that $T_1 \simeq 396(r/\mathrm{AU})^{-1/2}$ K,

$$T = (396^4\, 290)^{1/5}\mathrm{K}\,(r/\mathrm{AU})^{-2/5}. = 352\mathrm{K}\,(r/\mathrm{AU})^{-2/5}.$$

At $r = 80$ AU the small grain has temperature $T \simeq 61$ K, which is significantly higher than the 44 K calculated for the 'big grain'.

ADDITIONAL NOTE:

This problem is motivated by the fact that astronomers are often faced with dust disks, in which size distribution of particles is such that most mass resides in the large particles but most area in the smallest ones, a few microns radius. Those small grains have smaller emissivity (absorptivity) at the typical $\lambda_{eff} \sim 30\mu$m following from Wien's law for $T \sim 100K$. The reason is that they are much smaller than the wavelength of radiation they emit, and in that case the coupling between light and matter is always much weaker.[1]

A smaller emissivity, combined with the same absorbed energy flux at optical/UV wavelengths (where the star outputs most energy), means that the small particle cools less efficiently than a 'big particle' and has to achieve a higher equilibrium temperature T to compensate with larger σT^4 a smaller Q_{abs}. This, in turn, has directly observable consequences, which make the small grains of dust so interesting.

Namely, the infrared images of 2 disks with indentically distributed particles, one with 'big' (compared with 30 μm) and one with small particles, will be observed as 2 similar disks in the scattered light, but as rather different images in the 10-100 μm infarred range, because of different equilibrium dust temperature. (Which disk will be more extended in the IR?) Comparing infrared and visible images with the models based on different sizes of particles, we can derive the dominant size of dust grains. It often is equal to 2-15 microns, which on Earth we would classify as fine dust. The lack of grains smaller than one or two microns is understood to be the result of radiation pressure force being stronger than gravitational attraction by the star, leading to blow-out of submicron-sized grains around stars like β Pictoris and Vega.

11 [1p.] Garbage disposal in orbit

During a space walk outside the international space station, is it safest to throw a piece of garbage vertically down toward the Earth below, up, forward, or backward along the orbit? Why?

SOLUTION

It is safest to throw the garbage forward along the orbit. Definitely not vertically down, because the throw velocity being tiny compared with the orbital speed, we'd modify the direction of velocity of the garbage but leave the speed practically unchanged. Therefore, both kinetic and potential energy of garbage would match the energy of the space station and the object may return (having the same period P) and strike the station (gently).

The choice of forward or backward throw modifies the speed, not the direction of velocity, which is good - the piece of garbage now has a larger/smaller total energy, thus also semi-major axis and period. I'd be more comfortable throwing forward, since that increases the orbital period and the object would return to my orbital altitude after one orbital period behind me. It's probably just my psychological preference...

[1] One of the manifestation of that is, for instance, the law of the blue sky, stating mathematically the physical fact that a particle with radius $s \ll \lambda$, like a molecule of air in the atmosphere, scatters much more blue than red sunlight passing through the atmosphere (the larger the difference between s and λ, the less interaction).

12 [4p] Kepler's laws beyond Neptune

As you can see in http://www.gps.caltech.edu/ mbrown/planetlila/moon/, on September 10th 2005 astronomers at the Keck Observatory on Mauna Kea, Hawaii, took a look at the then 10th planet 2003 UB313 (now called a dwarf planet Eris, Gr. for struggle). A new instrument allowed them to see details as precise as those seen from the Hubble Space Telescope. Eris is currently 97 AU from the sun. The images revealed that it has a moon in orbit around it! It's called Dysnomia (lawlessness). We know that Eris is about 1.25 times larger than Pluto, i.e. 2900 km diameter (assuming it is covered with material of similar albedo A=0.6 as Pluto), but we don't actually know if it is more massive than Pluto. For example, a snowball could be bigger than a rock, while still much less massive. Pluto is a combination of ice and rock. If Eris is mostly rock, it should be more massive than Pluto.

Determining the orbital distance and period of the moon, with a little help from Kepler's laws applied to the Eris-Dysnomia system, will allow us to measure the mass of the planet and to find out if it is pure ice (density less than 1.5 g/cm^3), pure rock (density more than 3.5 g/cm^3) or a mixture of the two. Explain how.

Knowing that the observed moon-planet separation in the sky, equal to 0".53 (0.53 arcseconds), is the semi-major axis of the moon's orbit, and the orbital period from several observations is $P = 15.8$ days, is Dysnomia icy, rocky, or of mixed ice+rock composition?

SOLUTION

The third Kepler's law states that a small satellite (Dysnomia is indeed 60 times less bright than Eris) has a period P and mean distance from the planet (semi-major axis) a that are connected via

$$(P/\mathrm{yr})^2 = (a/\mathrm{AU})^3 (M_\odot/m)$$

where m is the mass of the planet. In our case, a is 97 AU times the angular separation in radians, i.e., 2.49e-4 AU = 37380 km. The Kepler's law reveals the mass ratio of Eris to sun to be $\mu = (P/\mathrm{yr})^{-2}\,(a/\mathrm{AU})^3$. which corresponds to a mass $m = 8.25e-09M_\odot$ or 1.65e+22 kg, or 1.27 Pluto's mass. So, this means Eris is larger AND more massive than Pluto by about 25-27 percent. You can now easily show that they have a similar mean density, thus both consist of a mixture of ice and rock.

13 [2p] Focusing factor

A planet is embedded in a disk made of small planetasimals (planet-forming bodies that you can imagine as comets or asteroid-type bodies). All solid bodies are rocky and have the same mean density ρ, but differ in size: the large body has radius 500 km, and the small planetesimals radius 10 km.

If the interparticle velocity in the small body system is equal to their escape speeds from ther surfaces, what is the gravitational focusing factor b/R of a big body, and how many times larger is its growth rate than in the case of no gravitational bending of trajectories (assuming perfect sticking of encountered bodies)?

SOLUTION

The formula for gravitational focusing factor from Lectures, simplifies in case of escape speed of a planet being much larger than the impact velocity v (equal to the smaller escape speed from a planetesmal) to the form $b/R_1 \approx \sqrt{(m_1/m_2)(R_2/R_1)}$, where subscript 1 refers to the planet and 2 to planetesimal. With equal densities of two bodies, this further simplifies to $b/R_1 \approx \sqrt{(R_1/R_2)^3(R_2/R_1)} = R_1/R_2 \sim 50$. The area from which the planet sweeps planetesimals thus exceeds its cross secion 2500 times, and the growth proceeds 2500 times faster than without gravitational focusing.

14 [3p] Stable lawlessness?

A rough guide to orbital stability of a moon of a planet is that its semi-major axis does not exceed $(3/11)(\mu/3)^{1/3}$ times the sun-planet distance, or 3/11 of its Roche lobe radius. It is only an approximate criterion because the concept of Roche lobe is strictly defined only for planets on exactly circular orbits (which, in the case of Eris, is not true). That "3/11" condition is fulfilled by our Moon by a small margin (that means our Moon's orbit is close to Hill instability, being at 0.256 of the Roche lobe radius, while 3/11=0.2727).

Apply the criterion to the newly discovered Dysnomia (lawlessness), moon of Eris, using the data and intermediate results of problem 3. Compare the result with the situation of our Moon.

SOLUTION

Knowing mass ratio of Eris to sun, 8.25e-09, we find that it's "Roche lobe" is $(8.25\text{e-}09/3)^{1/3}\cdot 97$ AU = 0.136 AU, whereas the orbit of Dysnomia is $a = 0.000249$ AU wide (see problem 6.3), and thus much much smaller (very stable). Dynamically, the Earth's and Erin's moons are very different.

15 [4p] Cleaning the solar system

What is the radiative blowout radius of small dust grains around the sun? That is, which grains feel radiation pressure stronger than central gravity?

Assume density of material equal to that of water, and albedo equal zero (black body), use the fact that photon's momentum is its energy divided by the speed of light, and geometrical cross section as the relevant cross section (disregard any diffraction and the so-called resonant scattering effects that might occur if the particle's radius is similar to the wavelength of light).

16 [3p.] "Beam me up Scotty, there is no intelligent life here"

In this problem, you play the role of Capt. Kirk of the spaceship Enterprise (from StarTrek) who has to decide about the safe orbital spacing from the giant planet in the system of ρ Coronae Borealis (Northern Crown), a.k.a. ρ CrB. Your Commander finds the parameters: $a = 0.22$ AU, $m = 1.04 m_J$ (jovian masses; that's mass ratio 0.001); eccentricity - small. You want to start on an initially circular orbit inside the planet's orbit. Traffic laws of the Federation prohibit you from flying it on Hill-unstable orbits with the type and amount of cargo you haul.

On the other hand, from your experience you know that if you start on an orbit separated by more than 10 million km from a planet's path, your Chief Engineer Scotty won't be able to beam your crew down to that planet's moon. Will you be able to beam your crew down to search for friendly life forms?

If you start from at angular separation of 180^o from the planet (on the other side of the star) at the minimum Hill-stable orbital separation, how much time is left until the first encounter with the planet? (That period is one-half of the synodic period. Hint: think about the angular speed differences.) Enterprise crew needs at least 4 weeks to prepare for the trip to the moon. Will they have enough time?

SOLUTION HINT

$r_L = a(\mu/3)^{1/3}$, where $\mu = m/M_*$. The $\sqrt{12}r_L$ "legal distance" (ensuring Hill stability) is slightly less than 10 mln km. Encounters of two orbiting bodies happen every synodic period, which you can obtain from the difference of angular speeds of the two bodies (synodic period = $2\pi/\Delta\Omega$). Thus the inverse synodic period is the difference of inverse orbital periods of the bodies.

17 [3p] Which β for blowout

Demostrate that radiation pressure coefficient $\beta \geq 0.5$ is needed for a newly created debris particle to escape from the host star to infinity. Assume that the beta-meteoroid[2] was released with negligible relative speed from a parent body travelling on a *circular* Keplerian orbit with specific energy E. Take into account that radiation pressure force adds a potential energy of $\Delta E = +\beta GM/r$ per unit mass of a body. Use the energy equation for Keplerian orbits.

SOLUTION

Compute the total energy at the moment that the dust is born, taking into account the changed potential energy. You should get $GM(\beta - 0.5)/r$, otherwise you won't get $\beta = 0.5$ when that energy is zero.

18 [6p] Eccentricity of an alpha-meteoroid

Demonstrate that the eccentricity of an alpha-meteoroid particle (by which we understand a particle noticeably affected by radiation pressure, but gravitationally bound to the star), in terms of its radiation pressure coefficient $\beta = F_{rad}/F_{grav}$ is given by

$$e = \frac{\beta}{1-\beta}$$

Assume that the alpha-meteoroid was released with negligible relative speed from a parent body travelling on a Keplerian orbit. Use the specific angular momentum (designated as L or l) and energy E for elliptic orbits:

$$E = -GM/(2a)$$
$$L = \sqrt{GMa(1-e^2)}$$

and take into account that the radiation pressure force adds a potential energy of $\Delta E = +\beta GM/r$ per unit mass of a body.

HINT: There will be many places where the central mass M gets replaced by $(1-\beta)M$, after the radiation pressure is added to gravity. One exception in this problem is the initial speed. The rest is (almost) normal Kepler's 2-body problem.

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[2] This name was first used for small meteoroids leaving the solar system on hyperbolic orbits, detected near the Earth in the act of radiative blow-out.

19 [5p] Transiting planet HD209458b

A figure cited below shows precise photometry of the transit of planet HD209458b in front of its star. Its orbital period is 3.5247 days, and the transit light curve obtained by Hubble Space Telescope is given by (Brown, Charbonneau et al)

http://www.obspm.fr/encycl/papers/HST-HD209458.pdf

The following information about the star is available: it's a G0V type star with mass 1.05 solar masses. For solar-type stars one can assume that radius is proportional to the square root of the mass, i.e. it equals 1 solar radius (=0.696 mln km) times 1.025.

Your task is to:

* Find the percentage of the visible area of the star obscured in the middle of transit and therefrom the radius of the planet, neglecting limb darkening effect (assume that the star's disk has uniform surface brightness.) Express the planet's radius in units of mean Jupiter radius (7e4 km) and compare with the information you find on the Web. Hint: Start worrying only if you've got a result many sigma away from the results in the literature/on the web. (Recommended site: the French encyclopaedia of extrasolar planets at http://www.obspm.fr/encycl/cat1.html)

* Find the semi-major axis "a" of the orbit. How much time should should it take for a small test particle orbiting precisely at I=90 degrees inclination (i.e., orbit exactly edge-on) to transit the star?

* Sketch the trajectory of the planet across the circle representing the star, at some height intermediate between zero (equator of the star) and grazing encounter with stellar "north pole". Take into account the finite size of the planet, and sketch the configuration at the times when the planet makes first/last contact with the disk of the star.

* Compare the actual timing of the transit seen in the data with your I=90 point-mass-transit prediction. Calculate, using simple geometrical considerations, the height at which the transit happens, and hence the estimated inclination I of the planet's orbit. How does your estimate compare (how many sigma away?) with the best determination cited in literature?

* Consider the radial velocity measurements, according to which the star wobbles radially by +-98 m/s. Find the minimum mass of the planet, assuming circular orbit.

* Could HD209458b be a rocky planet? To find out, calculate the true mass of the planet, using your value of I, and the minimum mass obtained from radial velocity measurements. Compute the mean density of the planet. (For comparison, compute the mean density of Earth, mostly rocky Neptune, and Jupiter.)

* Interpret qualitatively the density difference between HD209458b and Jupiter. Why so much difference? (Hint: estimate the blackbody SURFACE temperature at the appropriate distance from the star, assuming 1.15 times the solar luminosity for HD209458. Can you suggest other possible reasons for the radius exceeding Jupiter's radius?)

[Do as much as you manage. This is a very extensive homework-like problem. Again, an exam problem would not be asking you so many question! But it may ask some restricted set of those questions...]

20 [2p] Sublime problem

Silicates like olivine $(Mg,Fe)_2SiO_4$ or pyroxene $(Mg,Fe)SiO_3$, constituents of granite, are typical minerals in interplanetary grains and the primitive solar nebulae (protoplanetary disk). Estimate the minimum radius at which they can survive in a solar nebula without evaporation, assuming that their evaporation temperature is 1800 K, and they obey the blackbody radiation laws.

Equally common in planetary systems are ices, mostly water ice H_2O which sublimates (goes from solid to gas at low pressure) at 150 K, and carbon dioxide ice CO_2, which sublimates already at 80 K. Where in the solar system are the sublimation zones of these compounds?

Repeat the above estimates (i.e., compute sublimation radii) in the vicinity of Beta Pictoris, which is 8.5 times more luminous than the sun.

SOLUTION

Apply the black-body temperature law. In case of beta Pic, temperature at a given radius will be $8.5^{1/4}$ times higher, and the sublimation will occur at radii which are $8.5^{1/2}$ larger than in our system.

21 [2p] A gap and an end of the belt

Asteroid belt between Mars and Jupiter extends from less than 2 AU out to a small group of bodies called Thule group, with semi-major axis $a = 4.28$ AU. The largest gap in the distribution of semi-major axes of asteroids (collectively known as Kirkwood gaps) is at $a \approx 2.5$ AU. Which orbital resonances (commensurabilities) are responsible for the stability of the Thule group, endangered by the proximity to Jupiter, and which for the 2.5 AU Kirkwood gap?

22 [1p] Prove that...

..a $\pm 4\%$ uncertainty in the value of Earth's albedo (roughly known to be $A \approx 0.4$) results in about ∓ 2 C uncertainty in its mean temperature. Thus, systematic variation of cloud cover by a few percent over a century might cause either a global warming or cooling of the climate.

HINT

When looking for relative variations, always take a logarithmic derivative of an equation. That is, take a ln of both sides and then a derivative, for instance over time. Then, if you wish, yu can remove dt from each side of the equation.

PS. The IPCC models of climate on Earth are bad at predicting how cloud cover will change in time, as a matter of fact they largely ignore cloud cover variations, whether due to sun-earth interactions or due to the climatic change itself. Thus, they are much less believable than often believed.

23 [4p.] Avalanche in a disk

An A-type star (twice as massive as the sun) is surrounded by a disk with optical thickness $\tau_\perp = 0.01$ extending from 1 ro 101 AU from the star. At the inner edge, the average lifetime of particles against dust-dust collision is short, much less than 1000 years.

Once two particles collide, they shatter into $N_\beta = 100$ sub-micron sized debris with radiation pressure coefficient $\beta = 4$. What is the final, asymptotic velocity that the debris will achieve after leaving the system? Is it sufficient to catastrophically shatter other disk grains, that is is it larger than 100 m/s?

The total optical thickness in its midplane is equal $\tau_r = 0.1$, and on any section of the disk with optical thickness $d\tau$, the probability of collision between one small debris particle with disk particles equals $d\tau$. Formulate the differential equation governing the growing number of debris, $N(r)$, in an avalanche of debris flowing out from the inner edge if the disk to the outer and beyond. Starting with 1 debris particle at the inner edge, how many particles will flow out at the outer edge?

SOLUTION

The specific energy (per unit mass) of a grain with $\beta = 0$ is $-GM/r$, and of a grain with nonzero radiation pressure coefficient, $-GM(1-\beta)/r$. If β suddently (during the collision producing the grain) becomes $beta > 1$, like in our example, then the particle has total energy $-GM(1-\beta)/r + (GM/2r)$, where the second term is the kinetic energy. This gives total specific energy $E = GM(\beta - 0.5)/r > 0$, which caused the particle to depart to infinity, and retain there a positive kinetic energy equal E, corresponding to speed $v = \sqrt{GM(2\beta-1)/r}$, or $\sqrt{2\beta - 1}$ times the Keplerian circular speed at the point of production of a particle. In our example, the particle escapes to infinity with a speed of $\sqrt{7}$ times a few dozen km/s, so a pretty high speed that can easily lead to shattering (or even evaporation) of similar particles, if impacted.

Consider a ring of optical thickness $d\tau$. The probability $d\tau$ of debris-disk collision multiplied by number $N_\beta = 100$ gives the number of debris added to avalanche in the ring, hence the equation take the form

$$dN = +N N_\beta \, d\tau$$

and has the following exponential solution (obtained by first dividing both sides by N, which is known as separation of variables):

$$N(\tau) = N_0 \exp(N_\beta \tau),$$

where $N_0 = 1$ is the inital number of debris. In our case, one particle eventually is joined by $e^{0.1\cdot 100} = e^{10} = 22000$ outflowing debris particles. As you see, the location of the outer edge of the disk (or even how precisely the optical thickness changed along the path) are unimportant.

24 [3p] Accretional heating of protoplanet

Assume that a rocky, Earth-like planet accreted solid bodies from the surrounding planetasimal disk and grew to 1 Earth mass, being 1AU from the sun. Typical collision velocity was 1.2 times the escape speed from the planet's surface at every stage of growth. Planet keeps a constant density $\rho = 4000$ kg/m^3 during the growth. The gravitational focusing factor was constant, and the rate of supply of material was proportional to planet's radius squared. The formation took 30 Myr. At what rate was the collision energy supplied to the planet? What was the equilibrium temperature of a blackbody (ideally emitting) surface of the planet (energy gain always balanced energy loss). Was the heating by solar radiation (assume a steady luminosity of 1 $L\odot$) more or less effective in heating the planet during formation time? Consider how the answers to the questions evolved as the mass and radius grew from an asteroid to an Earth.

SOLUTION HINT

If a body with mass m falls onto a planet having 1.2 escape speed, it releases $1.44mv_{esc}^2/2$ of energy. If a mass supply rate is (dm/dt) or $\dot{m}$ for brevity, the instantaneous heating rate is $1.44\dot{m}v_{esc}^2/2$. Normalization factor for $\dot{m}$ rate can be computed from the total mass accreted and total time of accretion; data given in the problem should be sufficient to find the answers, but care needs to be taken to take into account the varying radius and mass of the planet, affecting the surface escape speed and thus energy supply rate.

25 [3p] Runaway all the way to a giant

Runaway growth of planetesimals and planetary cores by mutual collisions and accretion ends at the following 'isolation' planet-star mass ratio

$$\mu_{iso} = 2^{9/2}3^{-5/4}(\pi r^2\Sigma/M_*)^{3/2}$$

where Σ is the surface density of planetesimals in the disk, and M_* the stellar mass.

Consider the minimum solar nebula disk with the density distribution of solids given by

$$\Sigma = 100\,\mathrm{g\,cm^{-2}}\,(r/\mathrm{AU})^{-3/2}.$$

Where in the disk do you expect rapid giant planet formation, i.e. growth of the planetary cores by runaway accumulation all the way up to the critical core mass for atmosphere instability that happens at the core mass $\sim 10M_E$? (one Earth mass equals $1M_E = 3\cdot 10^{-6}M_\odot$, one solar mass $M_\odot = 2\cdot 10^{33}$ g). Compare your result with the distribution of planets in the solar system.

HINT

Look at the table of results in .ppt lectures. The only difference will be that in the lectures a slightly more accurate representation of Σ was taken. You should conclude that in terrestrial planet zone the isolation mass was 100–1000 too small to allow giant planet formation.

26 Simple orbital mechanics: justify answers by simple calculations

Half of angular momentum of an orbit of eccentricity $e = 0.1$ is removed. What is the new eccentricity?

Is the circular orbit the one with largest or smallest angular momentum, among orbits of constant total energy?

How many times larger is the potential gravitational energy than the total energy of a body in a circular orbit?

Why can't we assume that in interaction of two planets the sum of their energies is always constant?

If the orbit expands very slowly due to the decrease of mass of the sun, what will be the final orbit size for the Earth be if the sun loses 1/4 of its mass?

If there is a slight drag force on an artificial Earth satellite because of very some low density atmospheric gas along the orbit, then does that result in the velocity of a particle decreasing or increasing?

27 Extrasolar planet, Doppler detection

Resolving power of a very good spectroscope is 2 million, that is the ratio $\lambda/\Delta\lambda$, where λ is the wavelength of light observed and $\Delta\lambda$ is the spectral resolution (one pixel width in the spectrum). What resolution in velocity of

Doppler-shifted objects does such a spectrograph provide? Give the answer in m/s and compare with accuracy of extrasolar planet detection. Do you know why the two numbers are so different?

SOLUTION

$R = \lambda/\Delta\lambda = c/\sigma_v$, where c is the speed of light and σ_v resolving power in terms of velocity. Numerically, $\sigma_v = c/R =$ 0.3e9 m/s /2e6 = 150 m/s, which is much parger than a few m/s achieved by observers. The trick is not that observers can build a spectrograph with much higher R, they cannot. It is that they observe thousands of absorption lines simultaneously, and the photon counts near every absorbtion line change ever so slightly when the object Doppler-shifts the spectrum by a small fraction of 1 pixel in a spectrum. The correlated change of photon counts in hundred thousand or so pixels is picked up by numerical analysis software. Its very impressive feature of modern hardware-software systems.

28 Extrasolar planet detection, transits

You want to be able to detect planets the size of Mercury around faraway stars. What relative accuracy of photometry do you need, i.e. $\Delta I/I$ detection threshold? Assume solar type host star.

29 Reduction of pericenter distance for an Oort cloud comet

Consider a comet like comet Debiaski from movie "Don't look up" that is part of Oort cloud. At an apocenter point, the comet's speed is suddenly reduced by a factor q (i.e. the new speed v_a equals $v_a = q\,v_{a0}$. Knowing that before and after the perturbation eccentricties e_0 and e, correspondingly, were both close to 1, calculate by what factor does the pericenter distance of the elliptical orbit decrease.

SOLUTION

We know the expression for the aphelion speed:

$$v_a = \sqrt{(1-e)/(1+e)}\,sqrt GM/a.$$

Notice that this can be written, using aphelium distance $r_a = (1+e)a$ as $v_a = \sqrt{GM(1-e)/r_a}$. The new orbit comes back to the same aphelium distance as the old orbit (the point where the two orbits diverged), therefore $r_a = r_{a0}$.

Reduction of speed by a factor q leads to the relation between the old and new quantities (GM stays the same):

$$1-e = (1-e_0)q^2$$

The answer is

$$\frac{r_p}{r_{p0}} = \frac{a}{a_0}\,\frac{1-e}{1-e_0} = q^2\frac{1+e_0}{1+e}.$$

Since Oort cloud orbits are very elongated, $e \approx e_0 \approx 1$. This simplifies the answer to

$$\frac{r_p}{r_{p0}} \approx q^2$$

For instance, a reduction of speed to 1/4 of the previous aphelion speed results in a 16-fold decrease of the perihelion. This may make an unobservable comet visible. It could also destroy life on Earth, if it's on the new trajectory of the comet and giant planets don't deflect the comet sufficiently to save the Earth.

30 Also

Please look at 4 assignment sets and the preparation material for the midterm.

One or two out of 3 problems in the final exam is normally from the pre-midterm material.

We also have a nice page of tutorial problems this year. Look at those, including the suggested but unsolved problems.