ASTC25 (PLANETARY SYSTEMS) PROBLEM SET #1. DUE 1 FEBRUARY, THURSDAY, 1:05 PM

Points in the square brackets give the *relative* weight with which the problems count toward the final score. If you need any physical constants or planetary data, you may find them on the web and quote the values at the beginning of the solution. Follow the methodology of solving written problems explained in tutotial 1 & on the course web page, e.g., don't forget to check units.

[INTRODUCTION] **Background info on the 2-B problem**

Useful information about the 2-Body problem can be gleaned from the lecture notes, ch. 2 of Carroll+Ostlie, or our textbook by Lissauer and DePater. We summarize some important facts here, starting with **elliptic motion**.

The definition of the semi-major axis a (one-half the long axis of the ellipsis of *relative motion* of two bodies. Mass M in the equation of motion and all the resulting equations (incl. III Kepler's law) is the *total* mass ($M = m_1 + m_2$).

Apocenter (apoastron, apogeum, aphelion - depending on the name of the central body) and pericenter (periastron, perigeum, perihelion) are the names of largest and smallest length of the relative position vector **r**, or the minimum and maximum orbital separations.

The equation of an ellipse is $r = a(1-e^2)/(1+e\cos\theta)$, where θ is called the true anomaly, an angle between the body in orbit (end of vector **r**) and the direction to pericenter of the orbit. Parameter e, called eccentricity, obeys $0 \le e < 1$ and determines the elongation of the orbit from circle ($e = 0$) to needle-like very narrow ellipse ($e \to 1$). Actually, $e = 1$ describes a parabola and $e > 1$ a hyperbola (thus making the equation really much more general).

Specific angular momentum conservation reads $L = rv_\theta = r^2\, d\theta/dt = const$. In motion on an ellipse the value of the constant is $L = \sqrt{GMa(1-e^2)}$. From this expression you can derive the apocenter and pericenter speeds using $r = r_p = a(1-e)$ for pericenter and $r = r_a = a(1+e)$ at apocenter.

Energy conservation law reads $E = v^2/2 - GM/r = -GM/(2a) = const$. Here, E is the specific total mechanical (orbital) energy of a 2-Body system. By specific we mean 'per unit mass' of a test particle.

Escape speed is the minimum speed necessary to escape to infinity. It can be computed from condition $E = 0$, which is equivalent to saying that the particle is a borderline case of objects bound/unbound gravitationally to a star. $v_{\rm esc}^2 = 2GM/r$ at a distance r from the star.

Interestingly, the laws for **hyperbolic motion** are exactly the same. Hyperbolic motion has $E > 0$, $e > 1$, $a < 0$ (a is a negative parameter and we no longer call it semi-major axis, that name is meaningful only for ellipses). Notice that $a(1-e^2) = L^2/GM$ is still positive. The pericenter distance is still $r_p = a(1-e) > 0$, but apocenter distance is infinite (not defined). **Parabolic motion** has $e = 1$ and we can't use parameter a, we just use $r_p > 0$, $L > 0$, and $E = 0$, while the equation of parabola becomes $r = 2r_p/(1+\cos\theta) = (L^2/GM)/(1+\cos\theta)$.

1 [15p] Implication or Coincidence?

In 1650s R. Hooke and others have noticed that, according to Kepler's 2nd law, which using modern notation reads

$$L = rv_\theta = r^2\dot{\theta} = const.$$

the angular speed $\dot{\theta} = d\theta/dt = \frac{d\theta}{dt}$ falls with the inverse square of distance. They suspected that the same radial dependence is shared by the gravitational force. A hypothesis arose that the law of angular speed is a consequence of universal gravity law $f \sim 1/r^2$. Is this hypothesis correct, and why?

2 [35p] Eccentric speed

Towards the end of tutorial 1, we were in a hurry to derive the apocenter and pericenter speed from the angular momentum conservation law. We haven't really finished, and there was a mistake in the algebraic simplification of a formula. So...

Derive an expression for the speed at the pericenter and apocenter of an elliptic orbit with semi-major axis a and eccentricity e, in two separate ways, based solely on the principle of:

(i) Specific angular momentum conservation $L = const.$

(ii) Specific energy conservation $E = const$

Compare the two results and draw conclusion.

(iii) Apply the results to dwarf planet Sedna. Its perihelion is at $r_p = 76.19$ AU, and aphelion is very far from the sun, at $r_a = 937$ AU. Compute a and e. Compute the speeds at perihelion and aphelion in km/s and the perihelion/aphelion speed ratio. At perihelion, compare the speed of Sedna with the local escape speed (What kind of trajectory would result if the two were equal?). At least how much speed (in km/s) would we need to add at perihelion and how much at aphelion to eject Sedna from the solar system?

3 [30p] Exoplanet Kepler-10b

An exoplanetary system around an old, solar-like G2-type star with mass $M = 0.90M_\odot$, has a planet discovered in 2011 by a mitary telescope facility sharing data with civilian astronomers, located on the Haleakala volcano in Maui, Hawaii. It actually has more planets (find out how many!), but we'll talk here about the one called Kepler-10b. Its orbital plane is inclined by about 14 degrees to the line of sight of the star; astronomers refer to this edge-on viewing angle as inclination $i \simeq 76^o$. The planet causes transits (exo-solar partial eclipses) every $P = 0.8374907$ days.

Assuming a circular planetary orbit (which all the observations support), and a mass of planet equal $m \ll M$, using the concept of momentum conservation, calculate the formulae for: (i) the mean orbital distance (semi-major axis) of the planet a, (ii) the radius (in km) of stellar circular motion around C.M. that we'll call a_*, and (iii) the maximum observed radial speed of the star due to the planet, v_*.

Spectroscopic observations of absorption lines show that this radial motion of the star is in the range $v_r = \pm 2.37$ m/s, a very small quantity, but actually measurable with today's best spectrographs specially constructed to discover exoplanets. (They can even detect a walking speed variations of the star's center of mass, from a distance of 600 light-years!)

Evaluate a in AU and planet's mass m in the following units: kg, solar masses, and Earth masses (assume $M_\odot = 2 \cdot 10^{30}$ kg, $M_E = 3 \cdot 10^{-6} M_\odot$).

Transit timing data show that the radius of Kepler-10b equals 1.47 of the mean Earth radius. How many times denser than the Earth is the exoplanet? Is the bulk of the planet made of rocks, ice, or gases? (Hint: You won't need to know the value of Earth's radius to answer this question. But if you think you need it, it's 6371 km.)

Do you think life is possible on Kepler-10b?

4 [20p] Hyperbolic orbit of 1I/Oumuamua

The first interstellar object visiting our solar system was discovered in 2017 by a civilian/military telescope Pan-STARRS, located on top of the Haleakala volcano on the island of Maui, Hawaii. On 9 Sept. 2017, when that small elongated object was at the closest heliocentric distance (perihelion) $r_p = 0.2553$ AU, it had a heliocentric speed $v_p = 87.71$ km/s.

(i) Compare that speed with the local escape speed at perihelion, to show that 1I is an interstellar object, i.e. not bound to the sun gravitationally.

(ii) Derive and evaluate the eccentricity of the orbit ($e > 1$).

(iii) Derive parameter a of the hyperbolic orbit ($a < 0$).