ASTC25 (PLANETARY SYSTEMS) PROBLEM SET #2. DUE 15 FEB, 1PM

1 [20p] Earth and Moon

One billion years ago the day on Earth lasted $T_0 = 20$ hours (cf. Lecture 5). Given that the Earth's mass is 81 times larger than Moon's mass, and that the mean distance between their centers is now $a = 384400$ km,

(i) how far was the Moon 1 Gyr ago (find a_0 in units of a, and in km)?

(ii) how many days was the lunar month's length P_0 back then, and what is it now? (Calculate P, don't quote.)

Hints: On wikipedia, find and use the article about the non-dimensional moment of inertia. Angular momentum of Moon's orbit is the Moon's mass times the specific angular momentum that we usually talk about (cf. lectures). Neglect the small eccentricity of lunar orbit. Earth's radius is $R_E = 6371$ km, mass M_E =5.97e24 kg.

2 [35p] Poynting-Robertson effect

J. Poynting and later H. Robertson studied an interesting general-relativistic effect, which can be simply understood as aberration of light from a star, which causes radiation pressure vector to be tilted by about (v/c) radians, the ratio of circular speed to speed of light, with respect to the radial direction away from the star's center. (When a car travels, a vertically falling rain is seen on the side window as streaks inclined to the vertical at the angle whose tangent function equals v/c, where v is the car speed and c the speed of the falling rain. Read more on
`https://en.wikipedia.org/wiki/Poynting-Robertson_effect`

Radiation pressure falls off with the same second power of distance as gravity. A non-dimensional β coefficient contains all we need to know about it; β depends on the size properties of a particle. A gravel particle (4 mm diameter) can have $\beta \approx 0.01$, a fine sand particle (0.4 mm diameter) can have $\beta \approx 0.1$, and dust particles even larger values.

Aberration causes a slight tangential component of the radiation pressure force (acceleration, really):

$$\mathbf{F_{PR}} = -\frac{\mathbf{v}}{c}\frac{\beta GM}{a^2}$$

Here, a is the semi-major axis (we will only consider circular orbits, so we can replace distance r with a everywhere; in fact one can show that eccentricity is damped by the Poynting-Robertson drag force, which justifies that assumption). The force is directed against velocity vector $\mathbf{v}$, thus it decreases both energy and angular momentum of the orbital motion of the particle. This causes a very slow drift of the particle: a decreases.

(i) Using the dependence of orbital energy on semi-major axis in Kepler problem, show that the relative rates of change obey

$$\frac{\dot{E}}{E} = -\frac{\dot{a}}{a}.$$

Substitute $\dot{E}$ from work-energy relationship $dE/dt = \mathbf{F} \cdot \mathbf{v}$ and present a formula for the radial drift speed $\dot{a}$ due to Poynting-Robertson effect.

(ii) Do the same using the angular momentum formula applied to circular orbits. As an intermediate result show that

$$\frac{\dot{L}}{L} = \frac{1}{L}\frac{dL}{dt} = \frac{\dot{a}}{2a}.$$

(The dot above L is almost invisible.) Find $\dot{L}$ from the torque formula $\dot{L} = aF_{PR}$. Prove that both methods result in the same equation for $\dot{a}$.

(iii) The formula for instantaneous time derivative of the distance is an ODE. Separate variables and solve this differential equation, with the initial condition that at time $t = 0$ particle is at $a = a_0$. Prove that the rate of change of a^2 is constant, and that the drift becomes therefore more vigorous as the particle approaches the star. Find explicit solution for $a(t)$ and make a sketch/plot. What is the total time of the drift t_{PR}, after which the particle is guaranteed to collide with the star (of perhaps evaporate nearby). It should be proportional to a_0^2/β. Evaluate t_{PR} for a sand particle and a gravel particle discussed earlier, starting from $a_0 = 1$ AU from the sun. Is the P-R drift effective in dusting off planetary systems?

3 [30p] Io's tidal heating

The average outward heat flux due to tidal flexing from the surface of Io is $F_{tid} = 2.25$ W/m^2. It is caused by the eccentricity of the satellite's orbit around Jupiter by moons such as Europa and Ganimede, and by the viscous dissipation of tidal flexing of Io's body.

Compare that flux with the insolation (flux of solar radiation, or irradiation) calculated as the absorbed part of the solar energy flux times the cross sectional area, divided by the surface area of Io (not just its sun-lit side). Assume that 63 percent of incoming solar radiation is scattered and 37 percent absorbed. Draw conclusions as to what is heating Io more: tidal interaction with Jupiter or the irradiation by the sun.

If its surface cools down according to Stefan-Boltzmann law (please read about it on wikipedia if you are unfamiliar), and summing up the tidal and radiative (absorbed) fluxes, what is the expected mean temperature T of Io's surface? Compare T with the actual mean surface temperature given by wikipedia.

4 [20p] Gravity matters

A. The International Space Station has many microgravity experiments, in which objects are weight-less while in orbit. Compute acceleration of the ISS due to Earth's gravity in units of g and prove that microgravity is a misnomer. The ISS orbits 400 km above the surface of Earth. Explain what's going on: how can a body's weight disappear, while gravity is still acting on it?

B. The Sun is 390 times further than the Moon, and its $1/q = 1/(3 \cdot 10^{-6})$ times more massive than Earth. Derive and evaluate the ratio of the accelerations with which the Sun and Earth are pulling the center of mass of the Moon.

The ratio may be surprising to you! It gives rise to the question: Why is the Moon a satellite of Earth not Sun? In other words, why does it not depart from Earth to become a dwarf planet? Solve the paradox.