ASTC25 (Planetary Systems) Problem set #1.**SOLUTIONS**

Points in the square brackets give the *relative* weight with which the problems count toward the final score. If you need any physical constants or planetary data, you may find them on the web and quote the values at the beginning of the solution. Follow the methodology of solving written problems explained in tutotial 1 & on the course web page, e.g., don't forget to check units.

[Introduction] **Background info on the 2-B problem**

Useful information about the 2-Body problem can be gleaned from the lecture notes, ch. 2 of Carroll+Ostlie, or our textbook by Lissauer and DePater. We summarize some important facts here, starting with **elliptic motion**.

The definition of the semi-major axis a (one-half the long axis of the ellipse of *relative motion* of two bodies. Mass M in the equation of motion and all the resulting equations (incl. III Kepler's law) is the *total* mass ($M = m_1 + m_2$).

Apocenter (apoastron, apogeum, aphelion - depending on the name of the central body) and pericenter (periastron, perigeum, perihelion) are the names of largest and smallest length of the relative position vector $\mathbf{r}$, or the minimum and maximum orbital separations.

The equation of an ellipse is $r = a(1-e^2)/(1+e\cos\theta)$, where θ is called the true anomaly, an angle between the body in orbit (end of vector $\mathbf{r}$) and the direction to pericenter of the orbit. Parameter e, called eccentricity, obeys $0 \le e < 1$ and determines the elongation of the orbit from circle ($e = 0$) to needle-like very narrow ellipse ($e \to 1$). Actually, $e = 1$ describes a parabola and $e > 1$ a hyperbola (thus making the equation really much more general).

Specific angular momentum conservation reads $L = rv_\theta = r^2\, d\theta/dt = const$. In motion on an ellipse the value of the constant is $L = \sqrt{GMa(1-e^2)}$. From this expression you can derive the apocenter and pericenter speeds using $r = r_p = a(1-e)$ for pericenter and $r = r_a = a(1+e)$ at apocenter.

Energy conservation law reads $E = v^2/2 - GM/r = -GM/(2a) = const$. Here, E is the specific total mechanical (orbital) energy of a 2-Body system. By specific we mean 'per unit mass' of a test particle.

Escape speed is the minimum speed necessary to escape to infinity. It can be computed from condition $E = 0$, which is equivalent to saying that the particle is a borderline case of objects bound/unbound gravitationally to a star. $v_{\rm esc}^2 = 2GM/r$ at a distance r from the star.

Interestingly, the laws for **hyperbolic motion** are exactly the same. Hyperbolic motion has $E > 0$, $e > 1$, $a < 0$ (a is a negative parameter and we no longer call it semi-major axis, that name is meaningful only for ellipses). Notice that $a(1-e^2) = L^2/GM$ is still positive. The pericenter distance is still $r_p = a(1-e) > 0$, but apocenter distance is infinite (not defined). **Parabolic motion** has $e = 1$ and we can't use parameter a, we just use $r_p > 0$, $l > 0$, and $E = 0$ while the equation of parabola becomes $r = 2r_p/(1+\cos\theta) = (L^2/GM)/(1+\cos\theta)$.

1 [15p] Implication or Coincidence?

In 1650s R. Hooke and others have noticed that according to Kepler's 2nd law, which using modern notation reads

$$L = r v_\theta = r^2 \dot{\theta} = const.$$

the angular speed $\dot{\theta} = d\theta/dt = \frac{d\theta}{dt}$ falls with the inverse square of distance. They suspected that the same radial dependence is shared by the gravitational force. A hypothesis arose that the law of angular speed is a consequence of universal gravity law $f \sim 1/r^2$. Is this hypothesis correct, and why?

SOLUTION

No, the hypothesis is incorrect. Kepler's areal law or the $d\theta/dt = L/r^2$ (L =const.) law does not follow from the universal gravity, despite having the same dependence on radius. It's a misleading coincidence.

To show this, we just need to review the proof of the area law inspired by Newton, in Lecture 3. It does not assume any form of attraction law, and would just as well work if gravity depended on some other power of distance, or non-power function of distance altogether. Conservation of angular momentum applies to all central force fields. That s the only requirement: force must always point precisely toward the center of attraction. Then $d\theta/dt \sim 1/r^2$.

Incidentally, some 1st-year university textbooks not long ago were wrongly stating that inverse-square law of gravity implies the speedup near pericenter. (I was asked to review one Canadian textbook and was paid today's equivalent of two grocery shoppings for pointing out such misunderstendings.) The books said angular speed obviously has to increase a lot near the center of gravity because the force increases sharply there. In reality, even a perfectly constant *central* force, well, even a force increasing instead of decreasing with distance, produces the exactly same rule: $d\theta/dt \sim 1/r^2$.

Some people cite Newton's Principia but haven't read the book. Well, to read the whole book is akin to torture today, as Newton may have deliberately written it in a complicated way to show his superiority to challengers, but the proof of the constant areal speed is exceptionally clear and could consist of only one original drawing with a caption: Look! (like some proofs by ancient Geometers).

2 [35p] Eccentric speed

Towards the end of tutorial 1, we were in a hurry to derive the apocenter and pericenter speed from the angular momentum conservation. We haven't really finished, and there was a mistake in the algebraic simplification of a formula. So...

Derive an expression for the speed at the pericenter and apocenter of an elliptic orbit with semi-major axis a and eccentricity e, in two separate ways, based solely on the principle of:

(i) Specific angular momentum conservation $L = const.$

(ii) Specific energy conservation $E = const$

Compare the two results and draw conclusion.

(iii) Apply the results to dwarf planet Sedna. Its perihelion is at $r_p = 76.19$ AU, and aphelion is very far from the sun, at $r_a = 937$ AU. Compute a and e. Compute the speeds at perihelion and aphelion in km/s and the perihelion/aphelion speed ratio. At perihelion, compare the speed of Sedna with the local escape speed (What kind of trajectory would result if the two were equal?). At least how much speed (in km/s) would we need to add at perihelion and how much at aphelion to eject Sedna from the solar system?

SOLUTION to (i)-(ii)

One can either use $E = -GM/(2a)$ or $L = \sqrt{GMa(1-e^2)}$, no need to invoke both in a proof.

The first solution uses energy. At peri/apocenter, which we will shorten to (p,a), the whole velocity is perpendicular to radius: $v = v_\theta$. Energy at those points can be written out as kinetic plus potential energy at the distance $a(1 \mp e)$ (upper sign for p, lower for a):

$$E = -\frac{GM}{2a} = \frac{v_\theta^2(p,a)}{2} - \frac{GM}{a(1\mp e)}$$

Solving for the speeds we get

$$v_\theta^2(p,a) = \frac{2GM}{a(1\mp e)} - \frac{GM}{a}$$

$$v_\theta^2(p,a) = \frac{1\pm e}{1\mp e}\,\frac{GM}{a}$$

$$v_\theta(p,a) = \sqrt{\frac{1\pm e}{1\mp e}}\,\sqrt{\frac{GM}{a}}$$

The second solution goes like this. At peri/apocenter, the whole velocity is perpendicular to radius: $v = v_\theta$. $L = rv_\theta$, at pericenter (apocenter) this becomes $L = a(1\mp e)v_\theta(p,a)$, from which

$$v_{\theta(p,a)} = \sqrt{GMa(1-e^2)}\frac{1}{a(1\mp e)},$$

which simplifies to

$$v_{\theta(p,a)} = \sqrt{GM/a}\sqrt{(1\pm e)/((1\mp e)}.$$

.

SOLUTION to (iii)

Sedna has the highest eccentricity among dwarf planets. (Eead more about its discovery, size, and the definition of a dwarf planet, in wikipedia.)

Sedna's semi-major axis is $a = (r_p + r_a)/2 = 506.6$ AU, which yields a highly eccentric orbit: $a(1\mp e) = r_{p,a}$, therefore $e = (r_a - r_p)/(r_a + r_p) = 0.8596$.

The minimum/maximum speed of Sedna is achieved at aphelion/perihelion:

$$v_{a,p} = \sqrt{\frac{GM}{a}}\sqrt{\frac{1\mp e}{1\pm e}}$$

. If $a = a_E = 1$ AU, then $\sqrt{\frac{GM}{a}}$ is the circular (Keplerian) speed at 1 AU, which numerically equals $v_{K,E} = 29.78$ km/s (it's good to remember at least the approximate value of 30 km/s! This saves some time in calculations, since constants like G don't have to be used.) If $a = 506.6$ AU, the circular speed is $\sqrt{506.6}$ times smaller.

$$v_{a,p} = v_E \sqrt{\frac{a_E}{a}} \sqrt{\frac{1 \mp e}{1 \pm e}} = 29.78\text{km/s} \sqrt{\frac{1}{506.6}} \sqrt{\frac{1 \mp 0.86}{1 \pm 0.86}} = \{4.640, 0.377\}\text{km/s},$$

Units check out.

Square of the local escape speed at perihelion, $v_{esc,p}^2 = 2GM/r_p = 2(GM/a)/(1-e)$. (At aphelion the minus sign is replaced by plus.) Comparing it with v_p^2 we see that the speed ratio is

$$v_{esc,p}/v_p = \sqrt{2/(1+e)} = 1.04.$$

(Computed in a analogous way, the ratio at aphelion is 3.65).

The minimum speed addition guaranteeing escape is 0.04*4.64 km/s = 0.18 km/s at perihelion, and 2.65*0.377 km/s = 1 km/s at aphelion. It will be much easier to do the addition at perihelion, not aphelion.

COMMENTS:

Criterion for an object to be on a parabolic orbit (which is barely able to escape from the solar system) is simply $v(r) = v_{esc}(r)$ or even simpler $E = 0$. It doesn't matter where the object is when you perform the test or where its velocity vector is pointing. It's enough to check whether one of the criterion is satisfied at one arbitrary point. This is because $E = 0$ condition is satisfied everywhere by the virtue of energy conservation.

3 [30p] Exoplanet Kepler-10b

An exoplanetary system around an old, solar-like G2-type star with mass $M = 0.90M_\odot$, has a planet discovered in 2011. It actually has more planets (find out how many!), but we'll talk here about the one called Kepler-10b. Its orbital plane is inclined by about 14 degrees to the line of sight of the star; astronomers refer to this edge-on viewing angle as inclination $i \simeq 76^o$. The planet causes transits (exo-solar partial eclipses) every $P = 0.8374907$ days.

Assuming a circular planetary orbit (which all the observations support), and a mass of planet equal $m \ll M$, using the concept of momentum conservation, calculate the formulae for: (i) the mean orbital distance (semi-major axis) of the planet a, (ii) the radius (in km) of stellar circular motion around C.M. that we'll call a_*, and (iii) the maximum observed radial speed of the star due to the planet, v_*.

Spectroscopic observations of absorption lines show that this radial motion of the star is in the range $v_r = \pm 2.37$ m/s, a very small quantity, but actually measurable with today's best spectrographs specially constructed to discover exoplanets. (They can even detect a walking speed variations of the star's center of mass, from a distance of 600 light-years!)

Evaluate a in AU and planet's mass m in the following units: kg, solar masses, and Earth masses (assume $M_\odot = 2 \cdot 10^{30}$ kg, $M_E = 3 \cdot 10^{-6} M_\odot$).

Transit timing data show that the radius of Kepler-10b equals 1.47 of the mean Earth radius. How many times denser than the Earth is the exoplanet? Is the bulk of the planet made of rocks, ice, or gases? (Hint: You won't need to know the value of Earth's radius to answer this question. But if you think you need it, it's 6371 km.)

Do you think life is possible on Kepler-10b?

SOLUTION

First, let's use the known period together with the generalized Kepler's 3rd law in the form

$$P = 2\pi\sqrt{a^3/GM}$$

or equivalently

$$(P/1yr)^2 = (a/1AU)^3(M_\odot/M).$$

It's called a generalized law because star's mass M other than solar is properly allowed. We obtain planet's semi-major axis $a \simeq 0.01685$ AU. This planet is very close to its star!

In the frame of reference of the center of mass, momentum of the star is negative momentum of the planet (total being zero). From this, denoting stellar speed as v_*, m the planet's mass, we have

$$v_* M = v m$$

where $v = \sqrt{GM/a}$. Notice that the formula gives the speed of planet's motion relative to star, not with respect to C.M., but orbit around the C.M. and around the star practically overlap on account of $m \ll M$. By the same token, stricly speaking we should have $M = M_* + m$ but that's in practice the same as M_*. We simply don't know stellar masses to high enough precision, to distinguish M from M_*. It's up to you, if you want to carry on the extra m in addition to stellar mass, that's fine, but not required.

If v_* is observed by Doppler spectroscopy, then the planet's mass can be found from

$$m \sin i = M v_* / \sqrt{GM/a}.$$

Notice that $\sin i$ is needed to account for the inclinantion of the orbit w.r.t. line of sight (observed radial speed is equal to the orbital speed in case $i = 90$ degrees only, otherwise its is less than orbital speed).

Units check: both sides are in kg, OK. Numerically, $m = 0.M_\odot = 0.051 M_E \simeq 10^{26}$ kg.

Now we have a super-Earth exoplanet (3.26 times more massive) whose size is 1.47 times larger than our planet, hence the volume about 1.47^3 times larger. Thus the exoplanet's density (mass/volume) is about $3.26/1.47^3 = 1.03$ times larger than Earths's mean density. It's a super-Earth of practically same density as the Earth. For sure it is not a gaseous giant planet, but a rocky, Earth-like world, although much much hotter, on account of the extreme closeness to the star (60 times closer than Earth-Sun distance, temperature should be more than 1500 C on the sunny side), so - life forms are unlikely to exist.

4 [20p] Hyperbolic orbit of 1I/Oumuamua

The first interstellar object visiting the solar system was discovered in 2017 by a civilian/military telescope Pan-STARRS, located on top of the Haleakala volcano on the island of Maui, Hawaii. On 9 Sept. 2017, when that small elongated object was at the closest heliocentric distance (perihelion) $r_p = 0.2553$ AU, it had a heliocentric speed $v_p = 87.71$ km/s.

(i) Compare that speed with the local escape speed at perihelion, to show that 1I/Oumuamua is an interstellar object, i.e. not bound to the sun gravitationally.

(ii) Derive and evaluate the eccentricity of the orbit ($e > 1$).

(iii) Derive parameter a of the hyperbolic orbit ($a < 0$).

SOLUTIONS

(i) At $r_p = 0.2553$ AU from the sun, escape speed equals $v_{esc}(r_p) = \sqrt{2GM/r_p} = \sqrt{2}v_K(r_p) = 83.97$ km/s. (I chose to evaluate circular speed at r_p first, from the value of such speed at 1 AU, which I remember as 29.87 km/s (cf. below). You may choose to use 30 km/s as Earth's speed, it's easier to remember and still faily accurate. You can also just plug the values of G and $M_\odot$ from wikipedia to obtain the same result. Clearly, $v_p = 87.7$ km/s exceeds the escape speed of almost 84 km/s, therefore the trajectory must be unbound (hyperbolic).

(ii) $a(1-e) = r_p = 2.0066$ AU is one equation, but we need another one to compute the two unknowns, a and e. Let's use the specific angular momentum $L = const.$, which at pericenter looks like as $L = r_p v_p$, but in general obeys $L^2 = GMa(1-e^2) = GMa(1-e)(1+e)$ which can also be written as $GMr_p(1+e)$. Therefore,

$$r_p^2 v_p^2 = GMr_p(1+e),$$

or

$$e = \frac{r_p v_p^2}{GM} - 1.$$

Units check out ok (verify!); remember to use velocity in m/s not in km/s if you decide to plug in any numbers here.

We can avoid remembering G =6.674e-11 in SI, and $M = M_\odot \simeq 2 \cdot 10^{30}$ kg, if we remember the value of Keplerian speed at Earth's orbit: $v_K(1AU) = \sqrt{\frac{GM_\odot}{1AU}} = 29.78$ km/s,

$$e = \frac{r_p}{1AU}\left[\frac{v_p}{v_K(1AU)}\right]^2 - 1 = 0.2553(87.71/29.87)^2 - 1 = 1.201$$

We see that eccentricity is significantly higher than 1, the orbit is thus unmistakably unbound (hyperbolic).

(iii) Parameter $a = r_p/(1-e) = -1.270$ AU is correctly negative to remind you that for hyperbolic paths, a is not equal to any "axis" length.