ASTC25 (PLANETARY SYSTEMS) PROBLEM SET #2. SOLUTIONS

1 [20p] Earth and Moon

One billion years ago the day on Earth lasted $T_0 = 20$ hours. Given that the Earth's mass is 81 times larger than Moon's mass, and that the mean distance between their centers is now $a = 384400$ km,

(i) how far was the Moon 1 Gyr ago (find a_0 in units of a, and in km)?

(ii) how many days was the lunar month's length P_0 back then, and what is it now? (Calculate P, don't quote.)

Hints: On wikipedia, find and use the article about the non-dimensional moment of inertia. Angular momentum of Moon's orbit is the Moon's mass times the specific angular momentum that we usually talk about (cf. lectures). Neglect the small eccentricity of lunar orbit. Earth's radius is $R_E = 6371$ km, and mass M_E =5.97e24 kg.

SOLUTION

(i) Spin angular momentum of the Earth was reduced from $\xi M_E R_E^2 (2\pi/T_0)$ to $\xi M_E R_E^2 (2\pi/T)$ now, where Earth's nondimensional moment of inertia $\xi = 0.3307$, from wiki article. The difference was transferred to the orbital angular momentum of the Moon, which changed by

$$(M_E/81)\sqrt{GM_E}(\sqrt{a} - \sqrt{a_0}),$$

We have

$$\frac{162\pi\xi R_E^2}{\sqrt{GM_E}}(T_0^{-1} - T^{-1}) = \sqrt{a} - \sqrt{a_0}$$

Divide by $\sqrt{a}$ to get

$$\sqrt{a_0/a} = 1 - \frac{2\cdot 81\pi\xi R_E^2}{\sqrt{GM_E a}}(T_0^{-1} - T^{-1}).$$

Let's check the units: in S.I. the fraction has m^2 in numerator and m^2/s in denominator (specific ang. mom.), and that gets divided by s, so units are ok. Don't forget to convert all quantities to S.I.!

Plugging in the values, $a_0/a = 0.921$, so $a_0 \simeq 354000$ km.

(ii) A billion years ago the orbital period of the Moon (siderial moon month) was $2\pi\sqrt{a_0^3/(GM_E)} =$ 24.3 days, and it is now $2\pi\sqrt{a^3/(GM_E)} = 27.5$ days.

2 [40p] Poynting-Robertson effect

J. Poynting and later H. Robertson studied an interesting general-relativistic effect, which can be simply understood as aberration of light from a star, which causes radiation pressure vector to be

tilted by about (v/c) radians, the ratio of circular speed to speed of light, with respect to the radial direction away from the star's center. (When a car travels, a vertically falling rain is seen on the side window as streaks inclined to the vertical at the angle whose tangent function equals v/c, where v is the car speed and c the speed of the falling rain. Read more on
`https://en.wikipedia.org/wiki/Poynting-Robertson_effect`

Radiation pressure falls off with the same second power of distance as gravity. A non-dimensional β coefficient contains all we need to know about it; β depends on the size properties of a particle. A gravel particle (4 mm diameter) can have $\beta \approx 0.01$, a fine sand particle (0.4 mm diameter) can have $\beta \approx 0.1$, and dust particles even larger values.

Aberration causes a slight tangential component of the radiation pressure force (acceleration, really):

$$\mathbf{F_{PR}} = -\frac{\mathbf{v}}{c}\frac{\beta GM}{a^2}$$

Here, a is the semi-major axis (we will only consider circular orbits, so we can replace distance r with a everywhere; in fact one can show that eccentricity is damped by the Poynting-Robertson drag force, which justifies that assumption). The force is directed against velocity vector $\mathbf{v}$, thus it decreases both energy and angular momentum of the orbital motion of the particle. This causes a very slow drift of the particle: a decreases.

(i) Using the dependence of orbital energy on semi-major axis in Kepler problem, show that the relative rates of change obey

$$\frac{\dot{E}}{E} = -\frac{\dot{a}}{a}.$$

Substitute $\dot{E}$ from work-energy relationship $dE/dt = \mathbf{F}\cdot\mathbf{v}$ and present a formula for the radial drift speed $\dot{a}$ due to Poynting-Robertson effect.

(ii) Do the same using the angular momentum formula applied to circular orbits. As an intermediate result show that

$$\frac{\dot{L}}{L} = \frac{1}{L}\frac{dL}{dt} = \frac{\dot{a}}{2a}.$$

(The dot above L is almost invisible.) Find $\dot{L}$ from the torque formula $\dot{L} = aF_{PR}$. Prove that both methods result in the same equation for $\dot{a}$.

(iii) The formula for instantaneous time derivative of the distance is an ODE. Separate variables and solve this differential equation, with the initial condition that at time $t = 0$ particle is at $a = a_0$. Prove that the rate of change of a^2 is constant, and that the drift becomes therefore more vigorous as the particle approaches the star. Find explicit solution for $a(t)$ and make a sketch/plot. What is the total time of the drift t_{PR}, after which the particle is guaranteed to collide with the star (of perhaps evaporate nearby). it should be proportional to a_0^2/β. Evaluate t_{PR} for a sand particle and a gravel particle discussed earlier, starting from $a_0 = 1$ AU from the sun. Is the P-R drift effective in dusting off planetary systems?

SOLUTION

(i) Since $E = -GM/(2a)$, taking the natural logarithm and doing time derivative on both sides, we get $\dot{E}/E = -\dot{a}/a$, or

$$\dot{a} = -a\frac{\dot{E}}{E} = \frac{2a^2\dot{E}}{GM}$$

We now use

$$\dot{E} = \mathbf{F}\cdot\mathbf{v} = -\frac{\beta GM}{a^2}\frac{v^2}{c}$$

where v^2 is the circular speed squared, equal to GM/a. We obtain

$$\dot{a} = -2\beta\frac{GM}{ca}$$

Units check out ok in this ODE.

(ii) Since $L = -\sqrt{GMa}$ on a circular orbit, taking the natural logarithm and differentiating both sides we get $\dot{L}/L = \dot{a}/(2a)$. Substituting the specific torque $\dot{L} = aF_{PR}$ and simplifying we get

$$\dot{a} = \frac{2a^2F_{PR}}{L} = -2\beta\frac{GM}{ca},$$

the same equation as in (i).

(iii) Separation of variables:

$$a\,da = -2\beta\frac{GM}{c}\,dt$$

allows an easy integration

$$a^2 - a_0^2 = -4\beta\frac{GM}{c}t.$$

This form of an integration constant $-a_0^2$ on the left-hand side satisfies the initial condition $a(t=0) = a_0$. The solution has a graph that looks a little like ellipse where it hits the x-axis:

$$a(t) = a_0\sqrt{1 - t/t_{PR}}$$

where

$$t_{PR} = \frac{a_0^2c}{4\beta GM}.$$

Checking units in these formulae; everything's A-OK. The extremely rapid end-phase is due to the steep increase of the product of two rapidly increasing factors, $v(a)/c$, and $-1/a^2$, as $a \to 0$ near the star.

Numerically, let's evaluate the time of spiral drift of a gravel particle with $\beta = 0.01$ from 1 AU to the sun:

$t_{PR} = (150e9)^2(0.3e9)/[4(0.01)(6.674\text{e-}11)\ (2\text{e}30)]$ s = 1.264e+12 s $\approx$ 40100 yr.

The P-R time scale shortens by a factor of 10 to $t_{PR} = 4010$ yr in case of a sand particle with ten times bigger radiation pressure coefficient β. Neither of these times of removal of small meteoroids from the vicinity of the Earth's orbit are long compared to the age of the solar system; in fact the dusting off of our planetary system's inner region is very quick. It's a bit different (how much different?) in the outer solar system.

As a final comment, we have neglected the fact that circular speed for particles pushed out by radiation pressure force is somewhat smaller than the Keplerian value. In our problem this would be a small correction to a small effect, but in general we might replace mass M of a star by a mass effectively reduced to $(1-\beta)M$ by the addition of radiation push, to obtain a better accuracy for β of order 1.

3 [30p] Io's tidal heating

The average outward heat flux due to tidal flexing from the surface of Io is $F_{tid} = 2.25$ W/m^2. It is caused by the eccentricity of the satellite's orbit around Jupiter by moons such as Europa and Ganimede, and by the viscous dissipation of tidal flexing of Io's body.

Compare that flux with the insolation (flux of solar irradiation) calculated as the absorbed part of the solar energy flux times the cross sectional area, spread over the whole surface of Io (not just its sun-lit side). Assume that 63 percent of incoming solar radiation is scattered and 37 percent absorbed. Draw conclusions as to what is heating Io more: tidal interaction with Jupiter or the irradiation by the sun.

If its surface cools down according to Stefan-Boltzmann law (please read about it on wikipedia if you are unfamiliar), and summing up the tidal and radiative (absorbed) fluxes, what is the expected mean temperature T of Io's surface? Compare T with the actual mean surface temperature given by wikipedia.

SOLUTION

We will evaluate many quantities immediately, which is a departure from our typical procedure of solving a problem.

Insolation flux on a perpendicular surface is $F_{sun} = \frac{L}{4\pi r^2}$, where $r = 5.2$ AU and L is the solar luminosity ($L_\odot = 3.83\text{e}26$ W). The units are W/m^2, as can be seen from the formula. Substituting numerical values (1 AU = 149.6e9 m), we get $F_{sun} = 50$ W/m^2. This however is per meter-square of cross-sectional area of the satellite, or a perpendicular flat plate, not yet per m^2 of the spherical surface; they differ by a factor of 4 ($4\pi R^2$ surface of a sphere vs. πR^2 its cross-section.)

Per unit surface area of the satellite, the time-average solar heating is (50/4) W/m^2 = 12.5 W/m^2. Only 37 percent of this value, that is 4.63 W/m^2, gets absorbed and heats the satellite surface. Thus solar heating is *bigger* than tidal heating by a factor of 4.63/2.25 = 2 times.

Stefan-Boltzmann law reads $F_{emit} = \sigma T^4$ (flux emitted per unit area of Io's surface is proportional to fourth power of surface temperature). F_{emit} equals the sum of tidal and radiative heating fluxes, i.e. 4.63 W/m^2 + 2.25 W/m^2 = 6.88 W/m^2. Therefore T =(6.88/5.67e-8)$^{1/4}K$ = 105 K. The mean temperature according to wikipedia is 110 K, which is different from our estimate by only 5%.

4 [20p] Gravity matters

A. The International Space Station has many microgravity experiments, in which objects are weightless while in orbit. Compute acceleration of the ISS due to Earth's gravity in units of g and prove that microgravity is a misnomer. The ISS orbits 400 km above the surface of Earth. Explain what's going on: how can a body's weight disappear, while gravity is acting on it?

B. The Sun is 390 times further than the Moon, and its $1/q = 1/(3 \cdot 10^{-6})$ times more massive than Earth. Derive and evaluate the ratio of the accelerations with which the Sun and Earth are pulling the center of mass of the Moon gravitationally.

The ratio may be surprising to you! It gives rise to the question: Why is the Moon a satellite of Earth not Sun? In other words, why does it not depart from Earth to become a dwarf planet? Solve the paradox.

SOLUTION to A

Earth's gravity at altidude $h = 400$ km above its surface (Earth's radius $R = 6371$ km) is $[(R+h)/R]^{-2}g \approx (1-2*400/6370)g = 0.874g$, that is only 12.6% weaker than on the surface. Microgravity is a misnomer. In a frame that is accelerated (with acceleration a), we add $-a$ to any acceleration on the bodies (cf. PHYB54 mechanics). It's simply a difference of accelerations on an object and on ISS, that produces what we call weight. In ISS, this difference pretty much vanishes, hence the weightlessness.

SOLUTION to B

It's almost the same as in the ISS and a body inside it. Both are falling toward the Earth but the relative acceleration not very unimportant. Well, more important than in the previous point, since ISS really does not pull the body gravitationally, while the Earth is pulling the Moon.

Let us derive the ratio of accelerations, which is the ratio of attracting masses ($1/q$) divided by the ratio of distances squared (390^2):

$$F_{M\odot}/F_{ME} = a_{M\odot}/a_{ME} = 1/(390^2 q) = 2.19$$

In the subscript, we name the body (M,E,$\odot$) attracted first, and the body attracting second.

The Sun $\odot$ is pulling the Moon much stronger than Earth, yet the Moon does not care and stays around the Earth. The reason is that BOTH Earth and Moon are attracted by the Sun, which pulls strongly on their center of mass, forcing it to orbit around the sun in 1 year. What could, in principle, tear the Earth-Moon system apart is the DIFFERENTIAL acceleration the sun imparts to the bodies: the tidal acceleration is a good name for it.

The maximum tidal effect due to the Sun happens when the Moon is in conjunction or opposition with the Sun (i.e. on the Earth-Sun line). Let us call the Earth-Moon distance d and Earth-Sun distance r. The difference of accelerations

$$\Delta a = -\frac{GM_\odot}{(r \pm d)^2} + \frac{GM_\odot}{r^2}$$

can be Taylor-expanded with error of order $O(d^2/r^2)$ as

$$\Delta a = \pm 2 \frac{GM_\odot}{r^2} \frac{d}{r}.$$

The numerical coefficient 2 comes from the power 2 in gravity law $-1/r^2$.

As advertised, the tidal force is much smaller that Earth's pull on the Moon ($a_{ME} = GM_E/d^2$):

$$\frac{|\Delta a|}{a_{ME}} = 2 \frac{M_\odot}{M_E} \left(\frac{d}{r}\right)^3 = 2/(390^3 q) = 0.0112$$

so there is no paradox! Moon's motion relative to our planet is not affected strongly by the Sun.

We see that tidal effects *increase* with the distance from Earth's center as d^3, the cube of distance, and decrease with the cube of the distance to the perturber (r^{-3}). As a result, the relative perturbation of Moon's orbit is only $2/(390^3 q)$, not $1/(390^2 q)$ that the incorrect argument (the paradox) suggested, i.e. almost 200 times weaker.