ASTC25. PROBLEM SET #3. SOLUTIONS

The first two problems use concepts introduced by French scientist Edouard Roche (1820-1883): the Roche lobe and the Roche limit of tidal disruption.

1 [25p] Can you believe it?

Having observed a transiting giant exoplanet with a short orbital period $P = 5.87$ days, circling around a star of mass $M = 0.79 M_\odot$, your friend, a graduate student of astronomy, reports that in the lightcurve he sees a signature of an occasional eclipse of a small moon orbiting the planet. Before he announces the details in a press release, your friend asks you to help establish the plausibility of the discovery by showing that the orbit of the exo-moon is stable.

Compute the minimum (critical) heliocentric radius a_p and the orbital period P of a planet having a mean density $\rho_p = 0.95$ g/cm^3 (a value following from transit observations of the exoplanet, more than the density of Saturn but less than Jupiter's), which allows it to have a moon. Draw conclusions about the plausibility of the discovery of the moon.

Assume that only the moons with semi-major axes a closer to the planet than 3/11 of planet's Roche lobe radius are orbitally stable. (For instance, you can check that our Moon circles the Earth at about 1/4 of the Earth's Roche lobe radius.) In your calculation, assume that $M + m_p \simeq M$ to simplify the expression for Roche lobe radius (m_p is the mass of the planet.) Simultaneously, natural satellites held together by self-gravity must reside outside the Roche limit. Assume it is equal to 2.44 R_p, i.e. 2.44 physical radii of the planet.

SOLUTION

The allowed range of a (moon-exoplanet distance, i.e. distance of their centers) satifying both conditions of stability of the moon is given by

$$2.44 R_p < a < (3/11) a_p (m_p/3M)^{1/3}.$$

The range shrinks to a single radius and then disappears when a_p is small enough. We need to find that critical value of a_p, and the associated orbital period of the planet. It follows from equality

$$2.44 R_p = (3/11) a_p (m_p/3M)^{1/3}.$$

Writing m_p as volume times density: $(4/3)\pi \rho_p R_p^3$, we get

$$(11 \cdot 2.44/3)^3 = a_p^3 (m_p/3M) R_p^{-3} = a_p^3 \frac{4\pi\rho_p}{9M}$$

From this, we obtain the critical (or minimal) planetary a_p allowing it to have a natural satellite,

$$a_p^3 = (11 \cdot 2.44/3)^3 \frac{9M}{4\pi\rho_p}.$$

Interestingly, nothing in this answer depends on R_p, it cancelled.

We could plug that expression into the 3rd Keplers law and get the period immediately, but let's also find the minimum a_p explicitly.

$$a_p = (11 \cdot 2.44/3) \left(\frac{9M}{4\pi\rho_p} \right)^{1/3}$$

After checking the units (they're ok, since M/ρ has the unit of volume and $(M/\rho)^{1/3}$ of length) we can plug in all the values, either all in cgs (like a real astrophysicist :-) or all in SI units: $a_p = 0.945 \cdot 10^{10}$ m = 0.0630 AU. This corresponds to a period P_c given by generalized Kepler's law $(P/yr)^2 = (a/AU)^3 (M_\odot/M)$, or $P = 6.50$ days.

That is more than orbital period of the exoplanet in the discussed system, so the discovery of a moon is probably in error.

[What very unusual property of the moon would still allow the moon to be stable at $P = 5.87$ days? Think about the number 2.44 we assumed for Roche limit. Under what condition is it valid? (Cf. lectures!)]

2 [30p] UTSC students disprove a hoax, claim at least 11 moon months per year anywhere in the universe

ASSORTED PRESS. 1hr 25 minutes ago

TORONTO, Canada - A message from extraterrestrial civilization allegedly deciphered by TikTok user Nohan F. Knowes using a laptop connected to a parabolic satellite-tv dish, went viral world-wide. It was the second most-read news on internet on March 5, 2024. The first was titled "On Tuesday morning, Facebook, Instagram and Threads down. Thousands affected".

The message from E.T.'s, in addition to elaborate greetings addressed to world governments, informed that their civilization lives on a beautiful planet with one moon. They celebrate a holiday similar to our New Year that takes place in the last moon-month on their planet, which happens to be the 10th moon-month.

However, just one day after it started circulating, the message was shown to be a hoax by students from astrophysics course at the Scarborough campus of the University of Toronto. Students claim to have demonstrated that anywhere in the universe, there are at least 11 moon-months in a year. This astonishing fact has so far escaped the scrutiny of astronomers and raised a few eyebrows in academia.

The proof is reportedly based on the notion of Hill stability of the moon orbit, requiring it to be within 3/11 of the so-called Hill sphere, or Roche lobe radius of the planet, as well as on the centuries-old Kepler's 3rd law. The students mentioned the fact that our Moon is nearly Hill-unstable, orbiting at the distance of 0.256 (somewhat less than 3/11) of Earth's Roche lobe radius, and completing about 13 revolutions in a year. We eagerly await the promised publication of the proof.

Spurred on by their achievement, the students reportedly also estimated the *maximum* number of moon months in one year, anywhere in the universe. Details of the second proof were not immediately available, but X platfom commentators suggested that it may be based on the concept of "Roche limit" and an assumption that the mean density of a planet cannot exceed the density of iron.

Mr. Knowes could not be reached for comment by the time of this publication.

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Can you furnish the proof(s)? For full credit, provide the complete proof of the 1st claim (minimum number of months) without mistakes, or incomplete proofs of both claims (about minimum and maximum numbers of moon-months).

Take into account a subtelty of this problem. There are two different notions of a lunar month in circulation (pun intended): a sidereal and a synodic month. The number of those months in a year differ by exactly 1. More about this in `https://en.wikipedia.org/wiki/Lunar_month`. State which moon-month you are using in the derivation and express the minimum number of months using both.

SOLUTION, HINTS

This problem has two parts. Let us denote as m_p and a_p the planet's mass and distance from a star (assumed constant in time), while M is the star's mass. We can use the expression for Roche lobe radius $r_L = (\mu/3)^{1/3}\, a_p$, to obtain the marginally stable orbit size $r_1 = (3/11) r_L$.

The ratio of the period of motion around the sun to the (sidereal) period of motion around the planet is the minimum number of sidereal moon months in a year, $n = P/P_1$. From Kepler's 3rd law:

$$n^2 = P^2/P_1^2 = m_p a_p^3/(M r_1^3) = 11^3/3^2 = (12.16)^2\,.$$

Therefore $n > 12$. There are at least 12 sidereal moon-months in a year, or 11 synodic months (synodic months are longer).

Any result that is a universal constant like our result is beautiful, exactly because it does not depend on anything and applies anywhere. We get the same constraint no matter what star is circled by what planet, and at which distance, as long as there are stable satellite orbits. But dynamic destabilization of moons happens only around hot jupiters very close to a star, where a civilization would not survive the high temperature, so this is of no immediate concern to us or any other life forms.

As mentioned in the formulation of the problem, there is a nuance often overseen when solving such problems, and the minimum is 12 sidereal, or equivalently 11 synodic months. What is the difference? We normally use periods of motion in inertial frame. Those are the sidereal periods (literally: with respect to stars), and we have used them. But the lunar calendar month is actually defined as the time for the moon to return to a nearly collinear position with the planet and the star. (BTW, colinear and collinear are both allowable spelling of the word.) The star-planet line turns slowly in space, which requires some extra time to achieve moon's colinearity. Hence, the calendar or "colinearily" period is longer than the sidereal period of a moon. From the 17th century one calls this time a synodic month (synod = Greek for meeting, as in: meeting on one line). The number of shorter sidereal months in a year should be exactly 1 more than the number of synodic months, because the sun-planet line makes exactly one (additional) turn per year. If you are unsure, write down the angular speed of the moon in two different frames, one inertial and one rotating with sun-planet line of apsides). In the solar system, 13.37 sidereal lunar months, and 12.37 synodic (calendar) lunar months occur in a year. See the link given in the problem formulation. As a consequence, anywhere in the universe there are at least 11 (calendar) moon-months in a year. I think that E.T.'s, just like us, are using synodic months. But if they use sidereal months, the number is 12, so in any case it is larger than 11, which is a safer answer.

The second problem asks about the maximum number of lunar months. Instead of P_1 we should use $P_2 = P_2(a = 2.44R)$ or the period of motion of a satellite at the radius a equal 2.44 times the planet's physical radius R. This should give a higher number n, one which depends on the distance to the star. The usage of numerical coefficient 2.44 also requires some assumptions. So thus restriction lacks some of the beauty of the previous result.

3 [15p] A longer year?

Years should get longer, extremely slowly, because the sun gradually loses mass. So slowly that you may have a problem trying the compute the difference of orbital period P from one year to the next on a calculator or computer. For such occasions, we have calculus.

The sun loses mass at an average rate of $2 \cdot 10^{-14} M_\odot$/yr by ejecting its gas as the solar wind. Additionally, every year the sun loses mass m in thermonuclear reactions: $E = mc^2$, where c is the speed of light in vacuum, and E is energy radiated away in one year, which you can compute from sun's luminosity $L = 3.846 \cdot 10^{26}$ W.

By how many microseconds will the next year be longer than the current year due to solar wind, and how many due to both effects above?

Hint: Assume that the angular momentum and the eccentricity of Earth's orbital motion are conserved in both processes.

SOLUTION

Direct use of formulae, involving an attempt to subtract something like $2 \cdot 10^{-14} M_\odot$ from $M_\odot$ on a calculator is bound to be a numerical failure, due to a limited precision of the calculation. So let's use calculus.

Let $M_\odot$ by the current or starting mass of the sun. The constancy of the specific angular momentum of the planet reads

$$L = \sqrt{GMa(1-e^2)} = const.$$

(there are no torques on Earth in the process, and e=const.) makes this problem straightforward. Period P changes, because the semi-major axis and the central mass are both changing, in such a way that $Ma = M_\odot a_0 = const.$, where $M = M(t)$ is decreasing in time.

$$a = a_0 (M_\odot / M)$$

Orbital period equals

$$P = 2\pi/\Omega = 2\pi\sqrt{a^3/GM}.$$

(that's Kepler's 3rd law). Substituting for a from angular momentum conservation, we obtain

$$P = 2\pi (M_\odot/M)^2 \sqrt{a_0^3/GM_\odot} = (M_\odot/M)^2 \mathrm{yr}.$$

The derivative of a natural logarithm of both sides gives the relative change

$$\frac{dP}{P} = -2\frac{dM}{M}.$$

Numerically, the relative lengthening of the year due to solar wind is equal $dP/P = 4 \cdot 10^{-14}$. Given that $P = 3.1557\mathrm{e}7$ s, this is equivalent to $dP = 1.26\mathrm{e}{-6}$ seconds in one year (or 1.26 μs).

Next, $E = mc^2$. The sun loses $P \cdot L$ joules per year (where $L = 4\mathrm{e}26$ W), equivalent to $dM = -PL/c^2$ kilograms per year. The relative decrease is $dM/M = -6.73 \cdot 10^{-14}$, which is about (6.73/2) times larger than the loss due to the solar wind. Therefore, each year is longer than the previous by $1.26(1+3.365)\mu\mathrm{s} \approx 5.4\mu\mathrm{s}$. [There was a typo here in the solution initially posted. Sorry. I'll adjust your grades if you were marked down because of this!]

Incidentally, interactions between planets cause a much larger effect. Earth-sun distance dropped by 0.000003% per year recently. (The year is getting shorter, not longer.)

4 [20p] Dressing right for your trip to the Moon

The Moon on average is as far we are from the sun, so you'd expect a similar average temperature. But the truth is, due to the lack of atmosophere, temperature variations on the Moon are extreme. "Something I touched was hot - through the suit, so it must have been pretty hot" observed one Apollo astronaut after a Moon walk. In the shadow or during lunar night, the cold was so extreme that mechanical equipment was often malfunctioning because the grease froze solid.

Calculate $T(\theta)$, the lunar surface temperature as a function of sun angle above horizon. Assume that the sub-surface layer is insulating and conducts outward only a small thermal flux coming from the Moon's interior. That interior flux is a fraction q of the mean solar irradiation flux $F_\odot$, which on the Moon as on Earth is equal $F_\odot = 1373$ W/m^2. You can thus assume that the surface is heated only by the internal plus the solar flux (depending on sun angle θ) and cooled instantaneously by thremal re-emission that follows the Stefan-Boltzmann law. Efficiency of re-emission is 1, and of sunlight absorption is 1-A, where A=0.136 is the mean Bond albedo of the lunar surface, as measured by one satellite observatory. (The Moon has fairly dark rocks, they scatter into space only 14% of incoming light!)

Establish the unknown parameter q that gives best agreement of your $T(\theta)$ dependence with thermocouple temperature measurements of the surface by Apollo 17 astronauts, found in this picture: `https://www.workingonthemoon.com/a17psrf9-9.jpg`. Landing site on the 'shore' of the Sea of Tranquility was close (20°) to the equator and the sun was near the zenith ($\theta = 90°$) at noon. [Phase angle in the figure is a re-calibrated time, not exactly θ, but that's of no concern to you, because you are interested in temperature extrema only.]

What is the value of q? What are the minimum and maximum temperatures of the lunar surface in your model in K and °C? Plot or accurately sketch $T(\theta)$. Comment on all your results. For instance, what does q say about the amount of heat coming to the surface from the interior, relative to sun's heating?

SOLUTION

Solar angle reduces the maximum solar flux $F_\odot$ (on a perpendicular plate) to $F_\odot \sin\theta$. Energy loss = energy gain per unit time, so the flux balance reads

$$\sigma T^4 = (1-A)F_\odot \sin\theta + qF_\odot$$

$$T = (F_\odot/\sigma)^{1/4}[q + (1-A)\sin\theta]^{1/4}$$

Since the lowest observed $T_{min} \simeq 102$ K) should correspond in the model to $\theta = 0$ (no solar illumination), we get

$$q = \sigma T_{min}^4 / F_\odot = 0.00447$$

Factor $(1-A)\sin\theta + q$ in the temperature formula, together with $q \ll 1$ means that q affects the nighttime temperature ($\theta = 0$) but has very little influence on daytime peak T_{max}, when $\sin\theta = 1$ and $1 - A \gg q$.

Numerically $T_{max} = [(F_\odot/\sigma)(1-A+q)]^{1/4} = 380.8$ K, or 107.7°C.

The lowest modeled temperature is of course 102 K or -171.2 C, but this we've read off the picture and used as input, so it's not a prediction.

What we predict, however, is: (i) a value of T_{max} very high and very close to the measured value, and (ii) something astrophysically interesting, namely that the radio-isotope heat release inside the Moon is insignificant relative to the solar radiant heating (since we find $q \ll 1$).

BTW, now you understand why Apollo astronauts wore shiny, reflective, space suits (albedo 0.9). So in case the cooling fails on a mid-day Moon walk, they don't get burns.

In any case, after this exercise the prospect of living on a Moon base became less attractive to me. Not only have I already seen deserts on Earth, the quality of air is worse on the Moon (there is none), but I'd also hate to be subject to outside temperatures ranging from +108 to -171 Celsius when I walk my dog.