ASTC25. PROBLEM SET #4. SOLUTIONS, HINTS.

1 [30p] Radial velocity of HD 2039

In 2002, Paul Butler and Geoff Marcy, using radial velocity variations of the star HD 2039, have discovered a giant planet of 5 Jupiter masses orbiting on an fairly eccentric orbit.

1.1 Derive a general formula

for the radial velocity of the star of mass M, which has a planet of mass $m_p \ll M$ in orbit of eccentricity $0 < e < 1$, semi-major axis a, and a negligible inclination to the line of sight. The argument of pericenter ω is normally non-zero. We the observers are at nearly infinite distance on positive x axis. The periastron is deflected by angle ω from the line of sight.

Differentiate in time the equation of ellipse, and use the angular momentum conservation law

$$L = \sqrt{GMa(1-e^2)} = v_\theta\, r = const.$$

to derive both the radial (v_r) and the azimuthal (v_θ) components of planet's velocity relative to the star.

Hint: There are 3 angles useful for specifying the position of the planet on the ellipse. The first is the true anomaly θ, found in the standard equation of ellipse $r(\theta)$. Historical designation was f but it may be confused with force or acceleration, so you should perhaps stick with the name θ. The second is the mean anomaly (or mean position angle) nt, the historical designation being a confusing letter M; $n = \Omega = \sqrt{GM/a^3}$ is called the mean motion and t is time since pericenter. The third is the eccentric anomaly angle E (another confusing designation, eh?). Derived by Johannes Kepler in his book 'Astronomia Nova' (1609), it has the form:

$$nt = E - e \sin E.$$

[Historians have found that Kepler's equation was already formulated and solved in particular case some 750 years earlier by Habash al-Hasib al-Marwazi!].

The usual method of finding coordinates (r, θ) and from these where exactly the planet is at a given time t, consists of first multiplying time by n to get nt, then iteratively solving Kepler's equation to obtain eccentric anomaly E. You can start with the first guess $E = nt$ and iterate Kepler's equation like so: $E := nt + e \sin E$ [for an iterative solution of Kepler's equation, cf. his 'Epitome of Copernican Astronomy' (1621)]. You have to use numerical calculation, because there is no explicit formula for $E(nt)$, except for series expansions in the $e \ll 1$ case. Wikipedia tells you to iterate, but does not tell you how many times, in order to achieve good accuracy. If about 10 accurate digits of E are sufficient (& why would anybody need more?), from my experience the required number of iterations grows with eccentricity as $N_{iter}(e) \approx (5 + 10\,e)/(1-e)$.

There is a more efficient way to plot velocity curve as a function of time. You may create in Python a numpy array of E values uniformly covering the interval $0...2\pi$ or, for a nicer plot, $0...3\pi$ with, say, 800 points. Kepler's equation will convert it into an array of corresponding nt values without the need to solve anything iteratively. (In Python, algebra and trigonometry can be done element-wise on whole numpy arrays.) The only inconveniece may be that your grid of time values is not covering time axis in uniform intervals (unless $e = 0$). That is irrelevant for your present task, but if a uniform or some other non-uniform coverage of time axis is ever required (e.g. to compare with unevenly-sampled observation times), it may be obtained by numerical interpolation, which would still beat the speed of the usual, iterative method.

Finally, convert E to θ through identity

$$\tan\frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}}\,\tan\frac{E}{2}$$

and use $\theta(t)$ to obtain $r(\theta)$, angular speed $d\theta/dt = L/r^2$, and so forth. Remember that planet's orbit is turned by angle ω w.r.t. to the line-of-sight. Use trigonometry and dynamicsto derive the correct observed component of Doppler-shifted *stellar* velocity!

1.2 For planet b of HD 2039

adopt $a = 2.19$ AU, argument of periastron $\omega = 223^{o}$, star's mass $M = 1.2M_{\odot}$, planet mass $m_p = 0.005M_{\odot}$, and a high eccentricity $e = 0.68$.

Write your own script computing and plotting the component of star's velocity observed via Doppler effect. Express V_{obs} in m/s. *Please write your own script, where you comment every important line, stating what is being computed.* It is best to either put comments before or in the same line as code. Discussing the methods with fellow students is fine, but the implementation and computing must be your own.

For plotting, convert mean anomaly nt from radians to time in physical units by using Kepler's 3rd law giving period P of the planet (time from periastron in in the same units as P is equal $t = (nt)P/(2\pi)$.

The final plot of the $V_{obs}(t)$ (vertical axis in units of m/s) should have the horizontal axis labeled 'time from periastron [yr]', and a title including the name of the planetary system and your last name, or a least your initials. Submit the description and the plot to Quercus. Please do not submit Jupyter notebooks. If you use them, Jupyter can export them to PDF document, which you may separately submit to the assignments on Quercus as a 2nd file.

SOLUTION

In elliptic motion, the planet has at any time t the true anomaly θ, and velocity vector in polar coordinates equal to (v_r, v_θ).

Differentiation of $r(\theta)$ equation for planetary elliptical orbit gives $v_r = dr/dt$. After applying $d\theta/dt = L/r^2$ (ang. mom. conservation law) and cancellations:

$$v_r = e\sin\theta\,\frac{v_K}{\sqrt{1-e^2}}$$

where $v_K = \sqrt{GM/a}$ is the Keplerian circular speed at distance a.

The other component follows directly from L-conservation:

$$v_\theta = \frac{L}{r} = (1+e\cos\theta)\,\frac{v_K}{\sqrt{1-e^2}}.$$

What is the observed radial velocity of the planet? Let's place the observer on the x-axis at +infinity. The orbit (and all its velocity component vectors) must be turned turned ω radians in the counter-clockwise direction.

The radial part of velocity vector is at an angle $\omega + \theta$ to the x-axis, so we need to multiply the length of the radial velocity (v_r) of the planet w.r.t. its star by $\cos(\omega+\theta)$ to get the x-velocity of the planet, and by

$-(m_p/M_*)\cos(\omega+\theta)$ to obtain the contribution to x-velocity of **the star**. (The vector of stellar velocity is $-(m_p/M_*)$ times the vector of planet's velocity).

The azimuthal velocity component is at an angle $\omega+\theta+\pi/2$ to x-axis, and gives the following contribution to the x-velocity of **the star**: $-(m_p/M_*)\cos(\omega+\theta+\pi/2)v_\theta$.

Now the final nuance is that the observed radial velocity is defined as positive when the star has a negative x-velocity (runs away from the observer), not positive. It's conventional definition in astronomy. Which requires reversing the sign. Don't worry! If you've only made the overall sign mistake in V_{obs}, this will be forgiven by our TA.

Summing up the two contributions, and getting rid of the $+\pi/2$ in one argument of cosine by using the identity $\cos(\psi+\pi/2) = -\sin\psi$, we get the stellar, observed, radial velocity:

$$V_{obs} = -\frac{m_p}{M_*}\frac{v_K}{\sqrt{1-e^2}}\left[(1+e\cos\theta)\sin(\omega+\theta) - e\sin\theta\,\cos(\omega+\theta)\right]$$

Pretty complicated, eh? But Python loves such formulae.

The final chore is to express time not in radians, like phase nt, but in physical units, and evaluate the $v_K(a) = 29780/\sqrt{a/\mathrm{AU}}$ m/s. Period $P = (M_*/M_\odot)^{-1/2}(a/\mathrm{AU})^{3/2}$ is helpful here.

Here is the script solving Kepler's equation iteratively `https://planets.utsc.utoronto.ca/~pawel/ASTC25/Vr.py` and its results: `https://planets.utsc.utoronto.ca/~pawel/ASTC25/HD2039b.png`

You can see here that when we use the trick of not solving Kepler's equation iteratively but algebraically (sort-of in reverse), `https://planets.utsc.utoronto.ca/~pawel/ASTC25/Vr-noK.py` then the script is simpler and the graph stays exactly the same: `https://planets.utsc.utoronto.ca/~pawel/ASTC25/HD2039-noK.png`

2 [30p] A day at the extraterrestrial beach

`https://en.wikipedia.org/wiki/Ontario_Lacus` is the *other* Lake Ontario: a similarly shaped and sized lake on Titan, on of many lakes on Saturn's moon Titan, the largest of the 146 known moons of Saturn, and second-largest moon in the Solar System after Jupiter's Ganymede. It is the first extraterrestrial lake ever found and the first river on the western shore of the lake. The northern shoreline features ~1 km height hills, and wave-swept shoreline. The lake was discovered in 2008 by the Cassini spacecraft. It is filled with methane, ethane and propane (more or less a cigarette lighter fluid).

We want to go to the beach at Lake Ontario on Titan. Since Titan is about 9 AU from the sun, we need to dress well for the occasion: it could be much colder than here. You will compute the atmospheric temperature in an indirect way, from remote observations of density profile in the atmosphere.

(A) To establish the surface temperature, you will utilize the known structure of the dense, N_2-dominated, atmosphere of Titan (molecular mass $\mu = 28$, pressure 1.45 atm is significantly larger than in the N_2-dominated air in your room). Function $\rho(z)$ describing the vertical density profile of the atmosphere was obtained from remote observations and can be approximated near the ground by

$$\rho(z) = \rho_0\exp(-z/H)$$

where $H = 18.5$ km is the exponential scale height, and ρ_0 a surface density which, by the way, exceeds the atmospheric density on Earth.

Assume for simplicity that the atmosphere is isothermal, i.e. $p = (kT/\mu m_H)\,\rho = c_s^2 \rho$, where $T = const.$ is temperature, μm_H is the mass of the dominant molecule, and k = Boltzmann constant, $c_s = const$ is the isothermal soundspeed. Write and solve by variable separation an equation of the vertical hydrostatic force balance in the atmosphere. Gravitational acceleration is in local balance with gas pressure gradient acceleration, equal to $-(1/\rho)\,dp/dz$.

In computation of gravitational acceleration g_T near the surface of Titan, you can assume that it is constant in the whole atmosphere. Titan's mass is $M = 1.35$e23 kg = 0.0225 Earth's masses = 3.3 Moon masses, and its radius is $R = 2576$ km, which is 2576/6371 of Earth's radius (only 25% smaller than Mercury's radius.)

Prove that the solution indeed is an exponential function of height z, and write the relation connecting H to the constant temperature T near the surface. Then check the units and evaluate the temperature. [You should be concerned, if that temperature is not somewhere between 70 and 100 K, since these are the typical temperatures of satellites in the outer solar system.]

Knowing T, you now know how to dress for the beach-going. But more can be obtained from the observation of the atmospheric scale height.

(B) Write the equation for the temperature T_{eff} of a spherical body placed at distance r from the sun, having an ideal absorption and emissivity of $Q_{IR} = 1$ in infrared, but non-zero scattering coefficient or Bond albedo A in the visible. [Cf. problem set # 3]. Caused only partly by the reflecting surface, the albedo is mostly due to Titan's atmosphere, which is always hazy and scatters a significant fraction A of solar radiation back into space. Scattering dominates over retention of infrared radiation, so much so the atmosphere is called anti-greenhouse for its surface being colder than blackbody.

Using T computed in (A), estimate the mean albedo A of Titan. The Sun-Saturn distance during the observations was $r = 9.12$ AU. Google says that 1 AU = 149597871 km, I don't believe the last digits are correct :-) $L_\odot = 3.846 \cdot 10^{26}$ W. Any other data can be found in wikipedia.

SOLUTION

It is interesting to know how many Earth g's the gravity on Titan is:

$$g_T = GM/R^2 = g(M/M_E)(R/R_E)^{-2} = 0.1376\,g = 1.350\ \mathrm{m/s^2}$$

where we've plugged the data from the problem to evaluate the ratios.

Denote as $c_s^2 = kT/(\mu m_H)$ the square of isothermal soundspeed. The equation of ideal gas reads $p = c_s^2 \rho$. Hydrostatic equilibrium in z direction reads

$$-(1/\rho)\,dp/dz - g_T = 0$$

or after substitution of $p(\rho)$,

$$\frac{d\ln\rho}{dz} = -g_T/c_s^2$$

The r.h.s. is a constant with a unit of 1/length (check it!). Let's introduce a scale-height

$$H = c_s^2/g_T\,.$$

so that the r.h.s. of the differential equation becomes $-1/H = const$. Integration gives

$$\rho(z) = \rho_0\, e^{-z/H} \;,$$

where ρ_0 is the density at the surface. We see that the vertical profile of $\rho(z)$ and $p(z) = c_s^2 \rho(z)$ are indeed exponential, q.e.d. We know that $H = 18.5$ km, from which we can find the temperature:

$$T = \frac{\mu m_H}{k} g_T H$$

The units are ok. We can substtitute the known values, such as hydrogen mass $m_H = 1.6727\text{e-}27$ kg, $\mu = 28$, $k = 1.3806\text{e-}23$ in SI, and the end result is:

$$T = 84.7\; K$$

This reasonably-looking estimate of surface temperature can now be used to estimate albedo. Albedo A is present in the energy balance equation that we saw many times, for example in the lectures:

$$4\sigma T^4 = \frac{(1-A)L}{4\pi r^2}$$

$r = 9.12$ AU is the Sun-Saturn during observations. From this,

$$A = 1 - \sigma T^4 \frac{16\pi r^2}{L}$$

The fraction is inverse of the energy flux, and σT^4 is the energy flux, so dimensions are correct. It makes no sense to plug in the formula for T we obtained before, because it contains different constants that will not cancel with the constants in this equation for albedo. It is better to use $T = 84.7$ K directly. Numerically, we have σ =5.6703e-8 in SI, $L = 3.83\text{e}26$ W, and we find a very reasonable estimate of Bond albedo

$$A = 0.28$$

There is a slightly different definition of albedo called geometric bolometric albedo.
`https://www.sciencedirect.com/science/article/abs/pii/0019103574900360`
states that T = 84 $\pm$ 2 K (consistent with our T), and the bolometric geometric albedo of Titan is 0.21. For many years wikipedia was saying $A = 0.22$, presumably citing the Bond albedo of Titan, but now (2024) when you google ”albedo of Titan (moon)”, you get a much more satisfying answer: ”The bolometric Bond albedo, A , is then 0.27 ± 0.04, giving an effective radiative temperature Te= $84 \pm 2^{\circ}$K.” Even though Google should know that we never write kelvins as $^{\circ}$K, just K.

3 [25p] Janus and Epimetheus - corotating satellites

Two satellites, Janus and Epimethus were found in 1980 to orbit Saturn on almost identical, nearly circular orbits. Before the discovery of Epimetheus, only one satellite was thought to occupy Janus' orbit (discovered in 1966). On Dec 31, 2003, Epimetheus semi-major axis was $a_E = 151410 \pm$ 10 km and its orbital period was P = 0.694333517 days. Janus at the same time had the following orbit: $a_J = 151460 \pm$ 10 km, P = 0.694660342 days. Find the mass of Saturn, assuming the masses of both satellites are vanishingly small relative to the planet's mass.

Approaching very slowly, at small distance Janus and Epimetheus exert non-negligible mutual gravitational force, which modifies their orbits. The more distant satellite loses energy and angular momentum, which the less distant gains. Tis decreases the semi-major axis of the outer moon and increases the semi-major axis of the inner moon. Before they achieve conjunction, the moons cross the same value of semi-major axis, at which point they stop approaching each other along the orbit. The gravitational interaction continues, with the slower moon becoming faster and vice versa. After the swap, the corotational satellites start drifting apart, never being physically closer than $s = 10000$ km apart.

Sketch the motion of the satellites in one of the possible rotating frames of reference, for instance the frame of reference of the more massive Janus. Compute the *synodic period* P_{syn} of Janus and Epimetheus pair, defined as the period at which the two bodies and the planet repeat their configuration (this could be opposition, i.e., when the moons are in one line with the planet, on opposite sides). Based on it, predict which of the moons is closer to Saturn right now.

NOTE: `https://www.planetary.org/articles/janus-epimetheus-swap`. I mentioned the opposition and not the conjunction (moons in one line with the planet on the same side), which is normally used to define synodic period, because corotational satellites do not really have conjunctions. Even though s is a small distance compared with the circumference of their orbits, please take it into account in the calculation of synodic period: the moons' relative angular position changes by less that the full 360 degrees in one synodic period, because of the minimum distance s.

HINT: Use Kepler's relationship for angular speed Ω as a function of radius, and assume eccentricities are negligible. You may use expansion of any quantity like orbital frequency Ω or orbital period P with respect to radius. For instance, $\Omega(a+x) = \Omega(a) + (d\Omega/dr)x$, where $(d\Omega/dr)$ is evaluated at $r = a$, and equals $(d\Omega/dr) = (-3/2)(\Omega/a)$, in view of $\Omega \sim a^{-3/2}$). Here x is the difference of two similar a's. Alternatively, you can use the quantities a_J and a_E as they are given, and do arithmetic in double precision.

SOLUTION

As discussed in one of the tutorials, to talk about synodic period we need to first evaluate the difference of angular speeds of satellites Janus and Epimetheus. In fact we need the absolute value, since they swap places, while preserving the orbital separation, so $\Delta\Omega = \Omega_J - \Omega_E$ changes sign. Then we need to consider that in the frame of reference rotating with Janus (so that Janus is not changing the polar angle or azimuth, only changes the radius during encounters with Epimetheus), the trajectory of Epimetheus looks like a narrow horseshoe. The "missing part" of a full cirle has length $2s = 20000$ km, from the conditions of the problem. Subtracting appropriate portion of circumference $2\pi a_J$ we come to conclusion that instead of 2π, in time of one synodic period Epimetheus changes the polar angle by

$$\Delta\theta = 2\pi\left(1 - \frac{s}{\pi a_J}\right)$$

The synodic period is therefore less than $2\pi/|\Delta\Omega|$, namely it equals

$$P_{syn} = \frac{\Delta\theta}{|\Delta\Omega|}$$

What is $\Delta\Omega$? That we can obtain either using $\Omega_x = \sqrt{GM/a_x^3}$ for each satellite $x = J, E$ separately, or by writing

Taylor expansion of this Keplerian rotation equation and keeping the lowest order term only

$$|\Delta\Omega| \approx \frac{d\Omega}{da}|\Delta a| = \frac{3\Omega}{2}\frac{|\Delta a|}{a}$$

, where $|\Delta a|$ is the orbital separation of the pair (constant apart from brief exchanges). Units are ok, so we can continue. Here it does not matter which a we use; it would be a very small, higher order correction, if we were to distinguish a_J from a_E here, or Ω_J from Ω or from Ω_E. Let's use a_J and Ω_J then.

The final answer is obtained by substitution of $|\Delta a|$ and $\Delta\theta$ into the formula for P_{syn}, and remebering that $2\pi/\Omega_J$ is orbital period of Janus, P_J:

$$P_{syn} = 2\pi\left(1 - \frac{s}{\pi a_J}\right)\frac{2}{3\Omega}\frac{a_J}{|\Delta a|} = P_J\frac{2a_J}{3|\Delta a|}\left(1 - \frac{s}{\pi a_J}\right)$$

Numerically, I got $P_{syn} = 0.69466 * (2/3) * 3029 * (1 - 10000/\pi/151460) = 1373$ days, or 3.76 yr $\approx$ 4 yr. Synodic period is approximately 2000 times longer than the orbital period, due to extremely close orbits of the two coorbital satellites.

The problem formulation asked for Saturn's mass. But in the above derivation we never actually needed it. It can be gotten from formula for Ω^2, which is a Kepler's law applied to Saturn satellites: $GM = a^3(2\pi/P)^2$. Substituting the Janus' P_J (convered from days to seconds!) and a_J in meters, as well as $G = 6.674 \cdot 10^{-11}$ in SI, we get $M = M_S = 5.70551 \cdot 10^{26}$ kg.

This is correct, since we know (or should know) that Saturn is about 3 times less massive than Jupiter. What is the mass of Jupiter? We normally tend to approximate Jupiter-sun mass ratio as 0.001, and solar mass as 2e30 kg, but in fact Jupiter mass is not 2e27 kg but 1.89813e27 kg. That gives mass ratio relative to Saturn of 1.89813e27/5.70551e26 = 3.327. The inverse of this ratio is 0.30059. It turns out that it is much more (50 times more) precise to say that Saturn is 0.3 of Jupiter mass than to say that it is 3 times less massive than Jupiter.

4 [30p] Derivation of greenhouse effect

The subject of this problem is to model the greenhouse effect in a planetary atmosphere by assuming that it has optical thickness zero (is transparent) to starlight, but large in infrared thermal radiation (IR). Starlight thus heats the surface, which re-emits the entire obtained energy in IR. Before eventually leaving the planet, IR radiation is temporarily trapped in opaque atmosphere, which affects the temperature profile in the atmosphere and makes the ground warmer than in the absence of greenhouse gases.

The atmosphere will be assumed to transfer radiation only in vertical direction. It will be conceptually divided into an integer number τ of plane, horizontal layers, each able to efficiently absorb and re-emit at IR wavelengths. $\tau \geq 0$ is called the IR optical depth of the atmosphere. (Physically, absorption and re-emission of IR is done by greenhouse gases.) By this assumption, each layer is heating only two neighboring layers (one of which can be ground, or space, in case of the lowest/highest layer).

The ground and all the layers are in thermal equilibrium, i.e. at constant temperature T_n, where $n = 0, 1, ..., \tau$; index 0 describes the ground. The layers neither store energy nor cool down faster than they are heated by immediately neighboring layers (by layer 1 and the starlight, in the case $n = 0$).

Each atmosphere layer absorbs IR photons on upper and lower side and re-emits them in up and down directions. The ground only has one side, and layer number τ only absorbs on lower side but emits from both up and down-facing side equally.

The one-sided fluxes of infrared energy emitted by a surface at temperature T_n can be written as blackbody flux

$$F_n = \sigma T_n^4$$

(power per unit surface area). Consider a unit area of surface, so that you do not have to multiply all the fluxes by the same area (how large the area is does not matter for temperatures that we are seeking).

The ground receives some amount of starlight which for convenience will be written as flux $F_* = \sigma T_*^4$. The T_* can be considered a known quantity, although a particular value is not important here.

Your task is to find how the surface temperature $T_0(\tau)$ under an atmosphere of IR optical thickness τ depends on τ. How much higher will it be for $\tau = 1, 2, ...$, i.e. what are $T_0(1)/T_0(0)$, $T_0(2)/T_0(0)$, and $T_0(3)/T_0(0)$?

To find this, build a series of models for the atmospheres with optical depths $\tau = 0, 1, 2, 3$. The first model has 1 equation, describing energy gain and loss by the ground only, as $\tau = 0$ corresponds to no atmosphere. Case $\tau = 1$ is modeled by 2 equations (one for the ground and one for a single opaque layer), and so on. A pattern will become obvious after you do the first few models, allowing you to write an expression for the temperature valid for any τ. In fact, your result will be a good approximation for any real-valued τ as well.

HINTS

Make a sketch depicting the ground and all the layers of a given model, indicating by arrows energy transfer routes. It is tedious to write the associated fluxes σT_n^4, so why don't you label the arrows by the equivalent name, F_n, and also use F_n when you write the set of $\tau + 1$ energy balance equations. After you find the values of all F_n by algebra, you can express the final result as the ground temperature $T_0 = (F_0/\sigma)^{1/4}$.

SOLUTION

It is perfectly OK to follow the instructions in the problem to set up first 1, 2 then 3 equations describing the energy loss and gain in each layer. As long as after that exercise you have arrived at the conclusion that F's linearly grow in each model, starting from top layer that emits into space flux exactly equal to the flux the planet receives from the sun, proportionally to the number of the layer counted from the uppermost (renamed as 1), then you've got the right result. The temperatures grow as well, as 1/4-th power of this linear function, so you should have seen the surface being $(1+\tau)^{1/4}$ times hotter than T_*. Let's discuss and digress some more, maybe allowing some deeper insight.

The statement about temperature of the top layer being T_* follows from energy equilibrium of the planet as a whole: only the top layer radiates IR into space, which the whole planet receives flux that we agreed to call σT_*^4.

The statement that downward growth of F in each model must happen follows from consideration of energy balance in one of the intemediate layers with number n. Incoming energies are IR fluxes from neighboring layers $n-1$ and $n+1$; outgoing flux (two-sided cooling) is equal $2F_n$, so

$$F_{n-1} + F_{n+1} = 2F_n$$

or

$$F_{n-1} - 2F_n + F_{n+1} = 0\,.$$

Notice what happens when you have lots of layers. The above linear combination becomes better and better approximation to the second derivative $\partial^2 F/\partial n^2$ or $\partial^2 F/\partial z^2$, plus it says that the second derivative vanishes. This is becuase

$$\frac{\partial^2 F}{\partial n^2} \approx \left[\left(\frac{\partial F}{\partial n}\right)_{n+1/2} - \left(\frac{\partial F}{\partial n}\right)_{n+1/2}\right] \frac{1}{(n+\frac{1}{2})-(n-\frac{1}{2})}$$

while

$$\left(\frac{\partial F}{\partial n}\right)_{n+1/2} \approx (F_{n+1} - F_n)/1$$

$$\left(\frac{\partial F}{\partial n}\right)_{n-1/2} \approx (F_n - F_{n-1})/1$$

so that in the end on a grid of integer n's we approximate

$$\partial^2 F/\partial n^2 \approx F_{n-1} - 2F_n + F_{n+1}$$

as advertised.

Why is this whole math important? Because of the heat diffusion equation that contains this second derivative over spatial coordinate z (here represented by number n). In unsteady situation, heat equation says that the flux in each layer changes in time according to this PDE (partial differential equation)

$$\frac{\partial F}{\partial t} = const. \cdot \frac{\partial^2 F}{\partial z^2}$$

but in a steady (equilibrium) situation the time derivative is zero, and therefore relation $F_{n-1} - 2F_n + F_{n+1} = 0$ is satisfied (which simply says that whatever a layer gets from neighbors, it re-emits on two of its sides). So our layer-by-layer calculation is a discrete case of a more general, continuos, differential equation of energy diffusion, in the simple case of perfect thermal equilibrium.

If the second derivative of $F(n)$ or $F(z)$ is zero, then its first derivative over n or z is a constant, and that means that these functions are linear functions of their respective arguments. That is what you have obtained as a pattern emerging from a few low-N solutions, but as we see it is also valid in arbitrary, or even infinite, number of layers (such as is represented by the PDE of heat diffution). When we traverse the whole atmosphere, the whole optical thickness understood as an integer number of layers (excluding surface), on the surface of the planet we find (using $F = \sigma T^4$):

$$T_0 = (1+\tau)^{1/4} T_*$$

This formula is very similar, almost the same as a real radiation transfer theory derives, under certain assumptions. There, it is valid for a real, not only an integer τ; in this capacity it is very useful for the Earth and planets, as most have a fractional τ.