Preparation for midterm and other interesting problems. MANY SOLUTIONS INCLUDED.

ASTC25 (PLANETARY SYSTEMS) PREPARATION FOR MIDTERM EXAM. PART I = PROBLEMS.

There will be only one written problem in the midterm. It will not be very long, since you have to solve it in about 30 minutes. Some of the problems below are longer and more difficult, but contain diverese ideas and methods that may be helpful in exams.
Points in the square brackets give the relative weight with which the problems count toward the final score. It is way more important to provide a good path to solution, even if slightly inaccurate (due to approximations made), than to compute the final answer numerically. Please always check units before pluggin in numbers. Various physical constants are given at the end of this text. If you need additional constants or planetary data, ask google, and during the exam ask the instructor by raising your hand

1 Bolide, satellite and plane

You look up at zenith (straight over your head) after sunset and near each other you see: an airplane, the International Space Station satellite complex, and a bolide (big meteor burning upon the entry to the atmosphere). Calculate their angular speed in degrees per second (at zenith). Neglect the Earth's rotation speed in comparison with cosmic speeds.

For the ISS and plane, find the time of visibility above horizon in minutes, disregading any obstacles on the ground and refraction of light, but properly taking into accout the curvature of Earth. For the meteor, esimate the time to cross one radian arc in the sky.

Assume that the airplane moves at h=10.5 km, the bolide h=50 km, and the ISS at h=408 km above the ground, at constant h. Airplane's speed is v=870 km/h, and the bolide moves at 1.5 times the escape speed from Earth.

SOLUTION

Keplerian speed at h=0 is 7.9 km/s. The speeds are: 870 km/h = 0.242 km/s (plane), $\sqrt{GM/(R+h)} = v_K\sqrt{R/(R+h)} = 7.9*0.969 = 7.66$ km/s (ISS), and $1.5*\sqrt{2}*7.9$ km/s = 16.8 km/s.

Angular speed around the Earth is the linear speed divided by distance, which correspondingly, equals: 1.32, 19.3, and 1.1 degrees per second.

The bolide moves across 1 radian arc in $h/v \simeq 3$ s. For the airplane and the ISS, we cannot use the angular speed at zenith as if it were constant, since low above the horizon it will bve much smaller. Instead we should sketch the circles of the Earth and of the orbits, and a triangle made of points Z (zenith posion an orbit), O (center of Earth) and H (position on orbit seen from observation point on the horizon). Cosine of angle ZOH (we will call θ) is equal $R/(R+h)$, where $R = 6371$ km is Earth's mean radius, and h the altitude of circular orbit above the Earth.

Since the angle will be small (we will verify this afterwards), Taylor-expanding we have

$$cos\theta \simeq 1-\theta^2/2+O(\theta^4)$$

, where $\theta \ll 1$ is in radians. $O(\theta^4)$ reads "of order θ^4".

The inverse of the cosine accurate to order θ^2 is thus $1+\theta^2/2$, and at the same time equals $(R+h)/R = 1+h/R$. This gives

$$\theta = \sqrt{2h/R}.$$

Assumption that $\theta \ll 1$ was right, since $h/R \ll 1$.

The fraction of a circular path seen above horizon corresponds to angle 2θ, and in meters equals angle in radians times $R+h$, or $2(R+h)\theta$. This gives theoretical visibility time equal

$$t = 2(R+h)\theta/v = 2(R+h)\sqrt{2h/R}/v\,.$$

The the airplane is visible for $t = 7.1$ minutes (in practice less, since we can't see it just above the horizon at the distance of about 370 km), and the ISS is visible above horizon for up to 10.6 minutes. It is however, easy to track the plane and a satellite for many minutes after noticing them overhead. On the other hand, as mentioned, the bolide is much faster, it crosses 57 degrees in $\simeq 3$ s. The bolide would take minutes to cross the sky as a satellite, but it's much closer so it takes only seconds (and it can burn quickly).

2 Orbital migration at zero eccentricity

What is the orbital migration equation in Gauss perturbation theory if acceleration f is directed backward with respect to the velocity of a satellite orbiting around central mass M? Can you obtain the same result more directly, from the conservation of angular momentum? On what timescale is the orbiting body migrating? Hint: timescale is defined as $t_a := a/\dot{a}$.

SOLUTION

Gauss theory says $\dot{a} = 2f/n$, where the so-called mean motion n, otherwise known as mean angular speed Ω, equals $n = \sqrt{GM/a^3}$. The physical units agree, since an has units of speed, and $\dot{a}n$ of acceleration, just as f.

Angular momentum on a circular orbit of radius $r = a$ reads $r \times v_K$, so that

$$l = \sqrt{GMa}$$

. Under a small and steady, negative, acceleration f, the angular momentum is lost at a rate given by specific torque. $|d(r\times v)/dt| = |\dot{r}\times v + r\times\dot{v}|$ The first cross product is identically zero, $r = a$, and $\dot{v}$ is the acceleration f. Consequently, $\dot{l} = af$ =specific torque (pay attention to dot above l, it's hard to see!), i.e. torque divided by mass of the orbiting body. On the other hand, differentiating $\sqrt{GMa}$ over time, we have $af = dl/dt = \dot{l} = (1/2)\sqrt{GM/a}\dot{a}$, and substituting definition of n, $\dot{a} = 2\dot{l}/\sqrt{GM/a} = 2f/n$, as before.

Migration timescale is equal $t_a = a/\dot{a} = (1/2)na/f = v_K/(2f)$.

3 Solve the donkey paradox

Applying a small, **backward** directed, steady acceleration $f < 0$, we cause the orbiting body to **speed up** at a rate $d(v_K)/dt = -f > 0$, as if the force were applied forwards rather than backwards. The name of the puzzle derives from an alledged behavior of stubborn donkeys.

SOLUTION

Everythings ok. It's not a linear motion, and change of speed is not proportional to force (acceleration) along the trajectory. Specific angular momentum or the product of speed and radius diminishes. This causes the radius to diminish and the circular speed to gradually grow ($v_K\ a^{-1/2}$). From the previous problem we know that $\dot{a} = 2f/n < 0$; differentiating the speed formula $v_k = \sqrt{GM/a}$ and eliminating $\dot{a}$, we get as needed

$$\frac{dv_K}{dt} = (-1/2)\sqrt{GM/a^3}\,\dot{a} = (-1/2)\sqrt{GM/a^3}\,(2f/n) = -f.$$

4 When will the Hubble fall out of the sky?

Hubble Space Telescope is a 13.2 m 4.2 m (diameter) object with 11 ton mass, currently orbiting at $h = 535$ km above Earth. In how many years will it come down, if the atmospheric density is derived from the model of atmosphere above 25 km altitude given in https://www.grc.nasa.gov/www/k-12/rocket/atmosmet.html

Solve it either analytically or numerically, integrating over time the Gauss perturbation theory prescription for the rate of change of semi-majpr axis, $\dot{a}$. Consider that at 25 km height, the satellite is already practically destroyed (broken up, burned). Neglect Earth's rotation and winds, and satellite orbit's eccentricity. Assume cross sectinal area A equal to arithmetic average of the largest and smallest such area HST presents to the flow of rarified air. Drag force can be assumed equal to

$$F = -\rho A v^2/2.$$

Compare the result with the estimate of timescale for the sinking process given by he current value of $a/\dot{a}$.

SOLUTION

According to the upper atmosphere model, temperature of air is $T = T_1 + bh$, where $T_1 = 273.1 - 131.21$ K = 141.9 K, and $b = 0.00299$ K/m. Pressure is modeled as $p = p_0(T/T_0)^{-11.388}$, where $p_0 = 2.488e3$ Pa, and $T_0 = 216.1$ K.

The gas is ideal, i.e.

$$p = \rho T/(\mu m_H) = R_a \rho T$$

where $R_a = 287.1$ J/kg/K is the gas constant for air. Therefore, density $\rho = p/(R_a T)$, or

$$\rho = p_0(T/T_0)^{-11.388}/(R_a T) = p_0/(R_a T_0)(T/T_0)^{-12.388}.$$

Subtituting the $T(h)$ relationship we obtain $\rho(h)$ in the form

$$\rho = p_0/(R_a T_0)((T_1 + bh)/T_0)^{-12.388}.$$

We also need the expression for circular speed v of the satellite

$$v^2 = \frac{GM_E}{R+h} = \frac{v_K^2}{1+h/R},$$

where $v_K = 7.9$ km/s, $R = 6371$ km is Earth's radius, M_E Earth's mass, and G gravitational constant.

Drag force divided by object mass M equals acceleration $f(h)$:

$$f = -\rho(A/M)v^2/2.$$

$$f = -(A/M)(p_0/R_aT_0)((T_1+bh)/T_0)^{-12.388}\frac{v_K^2}{2(1+h/R)}$$

All the constants have been quoted above.

What's the solution of the Gauss perturbation equation with f substituted for tangential acceleration called T in Gauss' formulae? Let's first write the equation noticing that $a = R+h$ and therefore $\dot{a} = \dot{h}$. Any initial eccentricity e can be shown to decay quickly, so we set $e = 0$.

$$\dot{h} = 2\Omega f(h),$$

where $\Omega = v_K/R$ is the orbital angular speed, equal to $2\pi/P$, or 2π divided by orbital period. Substituting for f we get

$$\dot{h} = -\dot{h}_0((T_1+bh)/T_0)^{-12.388}(1+h/R)^{-1},$$

where

$$\dot{h}_0 = (A/M)(p_0/R_aT_0)v_K^2/2$$

5 [2 p.] Transfer orbit of the yellow convertible to Venus

Elon Musk sent one of his Tesla Model 3 convertible cars into space, as a test of Falcon rocket.

1. Design a transfer orbit to Mars-like orbit from the circular orbit of radius 1 AU, but nor from an immediate vicinity of Earth, i.e. disregarding both planet's gravity. It will involve the use of thrusters able to change the speed by a given amount, using appropriate amount of fuel.

How much extra linear speed and in which direction do you need, in order to give to the spacecraft so that it reaches Venus at pericenter (perihelion) of the orbit, and by how much & in what direction will you change the speed at pericenter, in order to put the spacecraft on a circular orbit there?

Compare these results with the additional speed to go from near-Earth orbit to Mars and circularize there.

SOLUTION

Part 1. Speed changes.

If we denote the planetary semi-major axes as $a_1 = 1$ AU and $a_2 = 1.52$ AU, and neglect planetary eccentricities, then the semi-major axis of the transfer orbit is $a = (a_1+a_2)/2 = 1.26$ AU. Its pericenter distance is $a_1 = a(1-e)$, hence $(1-e)a/a_1 = 1$ or $(1-e)(1+a_2/a_1)/2 = 1$. From this $e = 2/(1+1.52) - 1 = 0.2063$. The pericenter and apocenter speeds on this orbit are

$v_{p,a} = \sqrt{(1\pm e)/(1\mp e)}\sqrt{GM/a}\sqrt{GM(1\pm e)/a_{1,2}}$, (where M is the solar mass) because $a(1\mp e) = a_{1,2}$. Subtracting the local circular speeds we get

$\pm\Delta v = v_{p,a} - \sqrt{GM/a_{1,2}} = \sqrt{GM/a_{1,2}}(\sqrt{1\pm e}-1)$. It is easy to check that the units are correct. Substituting values and knowing that Earth's orbital speed is $\sqrt{GM/a_1} \simeq 30$ km/s, and therefore Mars' Keplerian speed $\sqrt{GM/a_2} \simeq 30/\sqrt{1.52} = 24.3$ km/s, we get $\Delta v = +2.95, +2.65$ km/s. (We need to add speed in the direction of motion at both points.) On the other hand, the extra speed needed to escape from sun's gravity is $(\sqrt{2}-1)V_K \approx 12.4$ km/s, which is almost an order of magnitude greater.

2. Fuel needed. (Extra difficult, midterm will not have this kind of problems)

How much mass of fuel do you need to add speed Δv to a spacecraft of decreasing mass equal to $M + m_f(t)$, where $m_f(t)$ is the decreasing fuel+oxidant mass and M the other, fixed mass of the spacecraft (structure, payload, equipment, etc.).

Assume that the exhaust gases escape from he nozzles at speed $v_f = 3.1$ km/s (which is valid for kerosine-based fuels); denote the infinitesimal decrease of spacecraft mass by dm_f, and the increase of spacecraft speed by dv. Consider the problem exactly, not as a rough estimate: formulate the equation of momentum change, where on the l.h.s. you write the infinitesimal change of the spacecraft momentum and on the r.h.s. the momentum flowing in the same time interval out of the nozzles with exhaust gases. Solve that equation to obtain Ciokowski or Tsiolkovsky (or the rocket) equation. With your solution in hand, you can apply for jobs at NASA as a rocket scientist.

What is the initial mass fraction m_f/M needed for the transfer orbit? For the escape orbit?

SOLUTION

Comment: there is a notion of specific impulse of fuel, which is very closely related to v_f. Namely, for kerosine-type fuels (jet fuel, hydrazine rocket fuel etc.) it is equal approximately $q = 320$ s. This is momentum you can obtain from unit weight of fuel (weight on Earth). Because weight is gm_f, this corresponds to $v_f = ga = 9.81$ m/s^2 $q \approx 3100$ m/s. For hydrogen + oxygen mixture it would be about $v_f = 4400$ m/s.

The differential equation of a single (say, last-) stage rocket is simply the conservation of momentum for the spacecraft+exhaust gases system:

$$[M + m_f(t)]dv + v_f dm_f = 0.$$

It can be solved by separation of variables:

$$v(t) = \int^v dv = -v_f \int [M + m_f(t)]^{-1} dm_f$$

After all the fuel and oxidizer are gone:

$$\Delta v = v_f \ln \frac{M + m_f}{M + 0}.$$

(the rocket equation) or

$$m_f/M = \exp \Delta v/v_f - 1.$$

Plugging in Δv = +2.95, +2.65, and +12.4 km/s, while $v_f = 3.1$ km/s, we get $m_f/M = 1.59$, 1.35, and as much as 54 (to reach escape speed - in that case obviously it is much better to use hydrogen, we'd get $m_f/M = 15.7$ then). But be careful planning a seemingly easy trip the nearby Mars. We've not considered the fuel to return back home!

6 Thermal temperature of orbiting gas falling onto accretion shock

[part A, 3p.] Derive an expression for the circular Keplerian velocity v_K of a body on a circular orbit from the centrifugal and gravitational force balance. What would be the kinetic energy E_k of a hypothetical particle (molecule) with mass μm_H, circling a star of 1 solar mass near its surface?

SOLUTION

$\frac{v_K^2}{r} = \frac{GM}{r^2}$ is the centrifugal (or the minus centripetal force) balanced with gravity, so $v_K = \sqrt{GM/r}$.

The energy would be $E_k = \mu m_H v_K^2/2 = GM_\odot \mu m_H/(2R_\odot)$, where $R_\odot$ is the star's radius.

[part B, 2p.] Suppose that all this energy was converted into heat (i.e., was thermalized, e.g. in an accretion shock at the surface) according to the expression $E_k = (3/2)kT$, (k is Boltzmann constant). What thermal temperature T would the gas made of such particles achieve in the post-shock layer?

SOLUTION $E_k = (3/2)kT = (\mu m_H/2)v_K^2$ and hence $T = GM_\odot \mu m_H/(3kR_\odot)$.

[part C, 1p.] Evaluate numerically that temperature for the sun.

SOLUTION:

D.I.Y. The expected temperature will be very high, compared to the surface temperature of the sun.

7 Stable or unstable

A dark molecular cloud core at a distance 100 pc from Earth has the angular diameter of 33 arcminutes. Its temperature is $T = 25$ K, and the estimated mass is $M = 207 \pm 22 M_\odot$. Is the core gravitationally stable?

SOLUTION

First, the angular diameter in radians is equal (33/60)*(180/π) = 0.0096, so the physical radius of the core, if spherical, is $R = 0.48$ pc.

Next, let's compute the Jeans mass (formula in the lecture notes):

$$M_J = \frac{kT}{\alpha m G}$$

where μm_H is the mass of H_2, the molecule of hydrogen, and α is some nondimensional number not much smaller than 1 – assume it's 0.25.

Then we get $M_J \approx 46 M_\odot$. Therefore, this molecular cloud core is Jeans-unstable and is expected to collapse gravitationally.

8 Simple orbital kinematics

What is the eccentricity and semi-minor axis of a heliocentric orbit, which spans the range of radii from 42 AU to 80 AU? What are the perihelion and aphelion speeds?

SOLUTION

Semi-major axis equals $a = (42+80)/2 = 61$ AU. Eccentricity e satisfies $80AU = a(1+e)$, hence $e = (80/61) - 1 = 0.311$. Semi-minor axis equals $b = \sqrt{1-e^2}\,a = 61\sqrt{1-e^2}$ AU, or $b = 58$ AU. Aphelion (perihelion) speed equals $v_K(a) = 30/\sqrt{a/AU}$ km/s times $\sqrt{(1 \mp e)/(1 \pm e)}$. Simplifying and substituting known values, the aphelion speed equals 30 km/s $\cdot \sqrt{(1-e)/80} = 2.78$ km/s. Find the perihelion speed yourself.

9 [4 p.] Simple Kepler 22b planet properties

What is the gravitational acceleration in Earth units, mean density, and the escape speed from the Kepler 22b planet (http://exoplanet.eu/catalog-transit.php)? What is the rough prediction for the black-body temperature on the planet, based on stellar data (luminosity $L = 0.79 L_\odot$)?

Calculate the expected mass if the planet has the same minerals and the same mean density as Earth, then increase the estimate of mass by 10% to approximately reflect the larger compression of rocks in the core and mantle of a more massive planet. (Only the upper limit on planet mass: $m < 31 M_{Earth}$ is known from Doppler spectroscopy.)

If there is water on the planet, is it fluid or frozen, assuming 1 bar pressure in the atmosphere?

Partial SOLUTION

The radius is 2.35 times the Earth's and the mass at equal density would be about $2.35^3 \sim 13$ Earth masses, so we should adopt 14 Earth masses to account for a larger compression of rocks and thus a 10% larger density.

10 [2 p.] Simple Kepler 22b humanoid properties

[This problem is really for fun, not really for preparation to midterm.There will be no task involving bio-mechanics.]

Suppose that human-like creatures live on Kepler 22b, which are simply rescaled humans (keeping all the building materials & proportions of the body the same). They must be rescaled overall, in order for their bones to be as resilient as ours, relative to their weight. The maximum load on a bone is simply proportional to its cross-sectional area. By what factor n would those aliens be rescaled?

SOLUTION

If we denote the length of humanoid body as h, the area of bones scales as $\sim h^2$ and the mass of a humanoid as $\sim h^3$. The weight is $(g/g_E)(h/h_E)^3$ times that on Earth, where g/g_E is the surface gravity acceleration normalized to $g_E = 9.81\text{m/s}^2$. In our case $g/g_E = 14/2.35^2 = 2.54$. The creature weight to bone strength ratio is $(g/g_E)(h/h_E)$ times that on Earth, and since it has to be the same, $n = h/h_E = 1/2.54$, the aliens are linearly 2.54 times smaller, in order to tolerate the 2.54 g_E acceleration on the surface.

11 [2 p.] Do they use Skype22 on Kepler 22b?

The humanoids on Kepler 22 b want to talk using a communications program Skype22 with their friends on the other side of the planet in real time (without noticeable delay). Speculate on whether they'd enjoy doing this like we do here on Earth, assuming the same kind of technology: electromagentic waves, optical cables, electronic equipment and so on, everything limited by the speed of light and the size of the planet.

As the speed of light is the same on Kepler 22b but its size is larger, communications will have to suffer larger delays. Meanwhile, a creature n times larger/smaller than an average human, will walk and talk (we assume these processes are evolutionary linked) slower/faster if the creatures are larger/smaller than us. Since we move our extremities roughly in sync with the swing of a pendulum the length of our extremities while walking (to save energy), let's assume that the speed at which creatures on different planets think and talk is proportional to the frequency of a pendulum their size. How much faster or slower do the aliens talk on Kepler 22b, assuming those scalings?

SOLUTION HINTS

The planet has a $n = 2.35$ times larger radius, so communications with the far away continents are n times more delayed. On the other hand, the frequency of a pendulum is proportional to $\sqrt{g/h}$, and hence $\sqrt{2.54 * 2.35} = 2.44$ times larger than on Earth. The aliens walk and talk 2.44 times faster, according to our assumptions, and think 2.44 times faster too (also the shorter body supports 2.44 times shorter impulse propagation time along neurons).

Meanwhile, the inevitable delays in Skype22 communications are 2.35 times longer. This results in a 2.44*2.35 = 5.97 times worse experience on Skype22, felt by a humanoid alien on planet Kepler 22b. (I think it would be unbearable. But I also think the same of Twitter etc. So, who knows.. :-)

12 [2 p.] Strange orbital mechanics

Show that the following strange result holds in the case when there is no aerodynamic drag anywhere:

Several bodies are released from a point outside the Earth with the same speed v but in different directions (such that they don't hit the planet). Gravitation of bodies other than the planet is taken to be zero. They will all come back to the launch point at exactly the same times, infinitely many times (unless they physically collide).

SOLUTION

The bodies have to come back to the release point, they just follow different elliptic paths. They will do this in the same time, if they have the same orbital period. Kepler's 3rd law says that the period depends only on the semi-major axis a and not on the elongation of the orbit. Orbital mechanics says that the total mechanical energy also depends on semi-major axis a only (energy per unit body mass equal to $E = -GM/2a$). The bodies do indeed have the same total energy E, since they have the same initial potential and kinetic energies per unit mass, which proves the constancy of the periods and the fact that they'll be meeting after 1,2,3,... orbital periods.

13 [2 p.] Mars: mountains and other features

Mars has radius = 3396 km, and mass = 0.107 Earth's, as compared to Earth's radius 6371 km.

[2p.]

Compute the mean density of Mars, both in absolute numbers (g/cm^3 would be an intuitive unit) and relative to that of the Earth.

SOLUTION

$0.107 * (6371/3396.)^3 = 0.706$ of Earth's mean density (and therefore quite similar chemically, just less compressed).

[2p.]

Assuming that the maximum tangent of the angle at which mountain slopes are inclined to the horizon is proportional to the local gravitational acceleraton, estimate the maximum slope on Earth from your experience, and using that estimate, on Mars (radius = 3396 km, mass = 0.107 Earths).

SOLUTION

Density relative to that of Earth is $(M/M_E)(R/R_E)^{-3}$ $0.107(6371/3396)^3 \sim 0.706$. The slightly smaller value is partially due to smaller a gravitational pull that compresses the rocks in the core, and perhaps also the different minerals.

[1p.] Compute the length of year on Mars in days.

SOLUTION

779.96 days from Kepler's 3rd law.

14 [2p.] Contact the rover

If you send a signal to a Martian rover, and it responds after 1.0 second of receiving the signal, then how long will you wait for the answer? Give precise minimum and maximum values.

SOLUTION: consider the minimum and maximum distances for all possible geometrical positions of two ellipses with given eccentricities corresponding to the two planets (find values of e in the wikipedia), divide distances by the speed of light.

15 [3p.] Orbital facts

If you brake the motion of a satellite on a Keplerian circular orbit by 1% of its speed, at the pericenter of the new orbit it will have an increased speed, instead of staying at a 1% slower speed. By how many % of the original circular speed? What if you increase the speed from circular by 1%? By how much do you change the size of the orbit in each of the above cases?

16 [4p.] Compute the tidal force

Find the tidal force pulling you apart, when you stand on the surface of a planet or another body with mass M and radius R. Compute the acceleration difference at your head and your feet, you may of course approximate the formula using the smallness of a human compared with the size of a planet. Express the answer in units of $g_E = GM_E/R_E^2$, where index E refers to the Earth, and evaluate that answer for the Earth's surface.

How small would a $M = 1M_\odot$ or sun-like mass body have to be in order to destroy yourself standing on the surface of that body? Assume the acceleration $a = 100g_E$ is enough to kill you.

17 [6p.] Chelyabinsk meteor explosion

Several years ago, a very big meteor entered the atmosphere and exploded over the Siberian city of Chelyabinsk. About 15 km above the city, it released energy equivalent to approximately 500 ktons of TNT (500e+6 kg of TNT).

A. What was that energy in Joules? (Find the equivalent energy of 1 kg TNT; find first what TNT is if you don't know!). If the Hiroshima bomb was rated at 12 ktons TNT, how much larger energy was released over chelyabinsk? (Give a computation, don't just quote wiki!). That certainly explains why more than a thousand people were injured by shards of glass.

B. If the meteor had a speed of 18 km/s, estimated from videos, before disruption, how much mass did it have and what size of a body was it, assuming stoney sphere with density 3 times as much as water density?

D. Assume the meteor's orbit had a perihelion at 0.8 AU, while its aphelion was in the astroid belt at 2.2 AU from the sun. Compute the speed of the meteor like so: From the formula for total orbital energy on the one hand and the instantaneous sum of kinetic and potential energies on the other hand, compute the the speed of the asteroid/meteoroid at $r = 1$ AU, but before it caught up with the Earth. Then take into account the depth of Earth's potential well (in terms of test particle speed, it's equal to the second cosmic speed or escape speed from Earth of 11.3 km.s) and adjust the speed up, because of Earth's gravity. What is that speed? After a vectorial subtraction of the Earth's speed, what relative speed is possible while entering the stratosphere? Draw a sketch of the orbit and the relevant speeds. Discuss your results.

[Chelyabinsk has been a subject of many jokes stressing the mythical thoughness of its residents. People of this city are all like Chuck Norris of the Siberia. It is said, for instance, that cosmonauts from Chelyabinsk on the Space Station step outside without spacesuits to smoke cigarettes. Also, the meteor did not disintegrate, it was a spaceship blown up by aliens when they realized which city they approach.]

18 [5p.] Two planets

Two planets circling clockwise the same star of mass M in the same plane (co-planar), interact gravitationally and modify their orbits in time. If they never approach closely, then the orbits evolve very slowly, changing orbital eccentricities and the angular momenta given by the formulae

$L_1 = m_1\sqrt{GMa_1(1-e_1^2)}$

where m_1, a_1, e_i are the first planet's mass, semi-major axis and eccentricity. Planet 2 has angular momentum L_2 given by an analogous formula with subscript 2. The total angular momentum of the system is conserved.

The orbits will precess, all the time keeping their initial energies and semi-major axes.

Suppose that one planet has the mass equal to $m_1 = 3$ Earth masses and at present has $a = 2$ AU and $e = 0.1$. The second planet has $m_2 = 1.5$ Earth masses, $a = 3.5$ AU and $e = 0.25$. What is the highest eccentricity that planet 1 can ever achieve? What is the highest eccentricity that planet 2 can ever achieve? What is the minimum distance ever between the planets?

SOLUTION

Let us denote the inital values by an added "0" subscript. $L = L_1 + L_2 = const$, therefore

$m_1\sqrt{a_1(1-e_1^2)} + m_2\sqrt{a_2(1-e_2^2)} = m_1\sqrt{a_1(1-e_{10}^2)} + m_2\sqrt{a_2(1-e_{20}^2)}$

Designating $c = \sqrt{a_2/a_1}\,(m_2/m_1) = 0.66144$, we have

$\sqrt{1-e_1^2} + c\sqrt{1-e_2^2} = \sqrt{1-e_{10}^2} + c\sqrt{1-e_{20}^2}\,.$

When one eccentricity increases, the other decreases. Maximum values of e_1 and e_2 (denoted with the "m") are obtained when the other planet is on circular orbit ($e = 0$):

$\sqrt{1-e_{1m}^2} = \sqrt{1-e_{10}^2} + c(\sqrt{1-e_{20}^2} - 1)\,.$

$\sqrt{1-e_{2m}^2} = (\sqrt{1-e_{10}^2} - 1)/c + \sqrt{1-e_{20}^2}\,.$

After substituting the initial values, we obtain the maximum eccentricities $e_{1m} = 0.2266$ and $e_{2m} = 0.2777$. It appears that the planets can remain in the oscillating interaction forever, since at these moderate eccentricities they never approach each other closely. To prove it, we evaluate the furthest astrocentric distance achieved by planet 1, which is $r_{1apo} = 2(1+e_{1m}) = 2.45$ AU (at which time planet 2 is at $r_2 = a_2 = 3.5$ AU). The minimum astrocentric distance to planet 2 is $r_{2peri} = 3.5(1-e_{2m}) = 2.53$ AU, when planet 1 is at $r_1 = 2$ AU from the star. Thus the minimum distance ever achieved is 0.53 AU.

Comment: The computed minimum distance is fairly safe for Earth-like planets. For instance, this is roughly the Earth-Mars minimum distance. If the planets had 3 and 1.5 Jupiter masses or more, however, the stability of the system would not be so clear. We would have to conduct a long-term numerical simulation to study it, since two Jupiter-class planets approaching to with 0.5 AU may modify their semi-major axes. In principle, the lighter planet 2 may then end up being ejected out of the system or thrown onto the star. At the very least, the system configuration may change.

19 [3p.] Conjunction of Venus and Jupiter, 2 March 2023

A close approach on the sky of Venus and Jupiter happened on the day of midterm of ASTC25. The distances from Earth were r1=1.36 AU and r2=5.76 AU (index 1 for Venus, 2 for Jupiter), while their distances from the sun were a1=0.732 AU and a2=5.20 AU. Mean radii of the planets are R1 = 6050 km, and R2 = 69900 km, their albedos A1 = 0.689 and A2 = 0.503. f1=0.57 of the full face of Venus was illuminated as seen from Earth, while the corresponding fraction for Jupuiter was equal f2=0.95.

Estimate the brightness ratio of the two planets during conjunction. (By brightness we mean the flux of energy received by a telescope on Earth.)

SOLUTION

TBA

20 Some possibly useful constants

Such a section will be found in exam texts.

If you don't have a calculator, state it in your solution and provide your calculation rounded off to 2 significant figures (numerical error less than $\sim$10% will not lower your score.) Otherwise, at least three significant figures should be carried.

$c = 2.99792 \cdot 10^8$ m/s, $= 2.99792 \cdot 10^{10}$ cm/s (speed of light)
$G = 6.67259 \cdot 10^{-11}$ m^3 kg^{-1} s^{-2} = 6.67259$\cdot10^{-8}$ cm^3 g^{-1} s^{-2} (gravity)
$k = 1.3807 \cdot 10^{-23}$ J/K $= 1.3807 \cdot 10^{-16}$ erg/K (Boltzmann)
$m_H = 1.66054 \cdot 10^{-27}$ kg $= 1.66054 \cdot 10^{-24}$ g (hydrogen mass)
$a = 7.5646 \cdot 10^{-16}$ J/K^4/m^3 $= 7.5646 \cdot 10^{-15}$ erg/K^4/cm^3 (radiation const.)
$\sigma = 5.67051 \cdot 10^{-8}$ J m^{-2} s^{-1} K^{-4} $= 5.67051 \cdot 10^{-5}$ erg m^{-2} s^{-1} K^{-4} (Stefan-Boltzmann)

$M_\odot = 1.9891 \cdot 10^{30}$ kg = $1.9891 \cdot 10^{33}$ g

$R_\odot = 6.9598 \cdot 10^{8}$ m $= 6.9598 \cdot 10^{10}$ cm

$L_\odot = 3.8515 \cdot 10^{26}$ J/s $= 3.8515 \cdot 10^{33}$ erg/s

1 AU = 1.496 $\cdot 10^{11}$ m = $1.496 \cdot 10^{13}$ cm

1 yr = 3.1558 $\cdot 10^{7}$ s; 1 pc = 206265 AU.

Earth's radius is $R_E = 6371$ km, Jupiter's mass is about 1/1000 $M_\odot$ or 316 Earth masses. Thus, Earth has $\sim 3e-6$ times the sun's mass.