Lyapunov and His Work

Group Project

Chad Fairservice (1008283956) &
Ali Jizzini (1008444015)

PHYD38

Department of Physical and Environmental Sciences
University of Toronto Scarborough
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1 Introduction

Few theorists in history have left as profound and far-reaching an impact across multiple disciplines as Aleksandr Mikhailovich Lyapunov. His pioneering work in stability theory laid the groundwork for fields such as celestial mechanics and chaos theory, while his contributions to probability theory remain fundamental in statistical physics and applied mathematics. From pure mathematics to engineering, Lyapunov's insights continue to shape our understanding of complex systems and their long-term behavior. His work serves as a reminder that even the most abstract mathematical ideas can lead to breakthroughs with lasting practical significance.

This report explores the historical and scientific significance of Lyapunov's life and contributions. We begin with a brief account of his intellectual journey, from his early years as a student to his emergence as one of the most influential mathematicians of his time. Next, we examine the development of his groundbreaking stability theory and its applications, particularly in nonlinear systems. Finally, we analyze the role of the Lyapunov exponent in characterizing chaotic behavior and present numerical calculations on a system of three-dimensional Hill's equations.

2 History

Aleksandr Mikhailovich Lyapunov was born on May 25, 1857, in Yaroslavl, located in present-day Russia. He was the eldest son in a family with great academic and artistic talent. His father Mikhail, was an astronomer at the Kazan Observatory, his brother Sergei was a composer, and his brother Boris was a member of the Russian Academy of Sciences.

Lyapunov demonstrated an early aptitude for mathematics and enrolled at the Faculty of Mathematics at the University of Saint Petersburg. His academic abilities led him to graduate the program with first-class honors two years earlier than the standard completion time. Upon graduation, he joined the Department of Theoretical Mechanics at the university, marking the beginning of his career in mathematics research.

2.1 Graduate Studies and Stability Theory

A major influence in Lyapunov's development and research direction was his mentor, Pafnuty Chebyshev, a mathematician renowned for his contributions to probability, statistics, and mechanics. For his master's thesis, Chebyshev proposed to Lyapunov a problem concerning the equilibrium shape of a rotating liquid. At that time it was known that a slowly rotating liquid in equilibrium assumes an ellipsoidal shape, and that at a critical rotation speed, this shape becomes unstable. The question posed was whether—at this critical velocity—the liquid transitions into a new equilibrium form that is only slightly different from the original ellipsoid. In modern mathematical language, this represents a bifurcation problem, and this problem plays a fundamental role in understanding planetary formation and stability.

Although Lyapunov's initial investigations were unsuccessful, he stumbled upon an equally profound question: are the equilibrium ellipsoids at small rotation speeds stable? This question marked the beginning of modern stability theory. Lyapunov's master's thesis, defended in 1885 at the University of Saint Petersburg, laid the foundations for the rigorous mathematical treatment of stability, which would later evolve into one of his most significant contributions to mathematics.

While studying at Saint Petersburg, Lyapunov met Andrei Markov, a mathematician best known for the development of Markov chains—a fundamental aspect of probability theory. This interaction had a great effect on Lyapunov's work, leading him to major contributions to probability theory, particularly in the study of central limit theorems.

2.2 Teaching and Doctoral Work

Upon completing his master's thesis, Lyapunov began teaching at the University of Kharkov. During this period, he published two groundbreaking papers on stability theory and made significant advances in central limit theorems. In 1888, he extended his research into the general stability of motion in mechanical systems with a finite number of degrees of freedom. His most influential work, *The General Problem of Stability of Motion*, published in 1892, introduced the modern abstract mathematical framework for stability analysis. Lyapunov submitted this work as his doctoral thesis to the University of Moscow and successfully defended it in September of 1892. The following year, he was appointed as a professor at the University of Kharkov.

Lyapunov's contributions were widely recognized, and in 1900 he was elected as a corresponding member of the Russian Academy of Sciences. By 1901, he became a full member and was appointed as the head of the Department of Applied Mathematics at Saint Petersburg—a position that had been vacant since the death of his mentor Chebyshev in 1894. Lyapunov's appointment again reaffirmed his status as a leading figure in Russian mathematics.

2.3 Chebyshev's Bifurcation Problem

Later in his career, Lyapunov returned to the problem Chebyshev had tasked him with years earlier, regarding the equilibrium shapes of rotating fluids. He managed to solve the problem and published *Recherches dans la théorie de la figure des corps célestes* (Research into the theory of the shape of celestial bodies) in 1903 and *Sur l'équation de Clairaut et les équations plus générales de la théorie de la figure des planètes* (On the Clairaut equation and the more general equations of the theory of the shape of the planets) in 1904.

Beyond his mathematical achievements, Lyapunov was widely recognized in academic circles. He was an honorary doctorate recipient from the universities of Saint Petersburg, Kharkov, and Kazan. He was also an external member of the Accademia dei Lincei in Rome, a corresponding member of the Paris Academy, and an honorary member of the Mathematical Society of Kharkov. His contributions to stability theory, probability, and dynamical systems continue to influence modern research in mathematics, physics, and engineering.

3 Contributions & Applications

Aleksandr Lyapunov made breakthroughs in several areas of mathematics, most notably stability theory, probability theory, chaos theory, and celestial mechanics. Among these, his work on stability theory stands out as a turning-point, dividing the field into a pre- and post-Lyapunov era. His doctoral thesis, *The General Problem of the Stability of Motion*, introduced a rigorous mathematical framework for analyzing the stability of dynamical systems.

3.1 Lyapunov Stability Theory

Before Lyapunov, stability in mechanics was understood through isolated results rather than a unified theory. Joseph-Louis Lagrange demonstrated that the equilibrium position of a conservative system is stable if its potential energy is a local minimum, however he could only prove this for systems with one degree of freedom. Dirichlet later generalized this result to higher-dimensional systems using a geometric proof based on the conservation of total mechanical energy. This approach extended Lagrange's result to systems with multiple interacting variables.

Lyapunov recognized that Dirichlet's method could be applied beyond mechanics. He introduced Lyapunov functions, which act as "energy-like" auxiliary functions that can be used to determine the stability of any dynamical system. His definition of stability—where small perturbations in initial conditions lead to only small deviations in motion—became the modern foundation of stability analysis.

In his 1892 doctoral thesis, Lyapunov introduced two methods for analyzing the stability of dynamical systems—a method of linearization and a method involving Lyapunov functions.

3.1.1 Linearization

Lyapunov's linearization method allows for the stability analysis of a nonlinear system by approximating it as a linear one near an equilibrium point. Consider the nonlinear system:

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x})$$

where $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{f}(\mathbf{x})$ is a continuously differentiable vector field. An equilibrium point $\mathbf{x}^*$ satisfies $\mathbf{f}(\mathbf{x}^*) = \mathbf{0}$. To analyze stability, we introduce a small perturbation $\mathbf{y} = \mathbf{x} - \mathbf{x}^*$, so that the system can be rewritten in terms of $\mathbf{y}$:

$$\frac{d\mathbf{y}}{dt} = \mathbf{f}(\mathbf{x}^* + \mathbf{y}).$$

Since $\mathbf{f}(\mathbf{x})$ is differentiable, we approximate it by a Taylor series expansion around $\mathbf{x}^*$:

$$\mathbf{f}(\mathbf{x}^* + \mathbf{y}) \approx \mathbf{f}(\mathbf{x}^*) + D\mathbf{f}(\mathbf{x}^*)(\mathbf{y}) + \mathcal{O}(|\mathbf{y}|^2).$$

Because $\mathbf{f}(\mathbf{x}^*) = \mathbf{0}$, and we can take $\mathcal{O}(|\mathbf{y}|^2)$ to be negligible, this simplifies to the linearized system:

$$\frac{d\mathbf{y}}{dt} = J\mathbf{y},$$

where J is the Jacobian matrix of $\mathbf{f}(\mathbf{x})$ evaluated at $\mathbf{x}^*$:

$$J = \left. \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right|_{\mathbf{x}=\mathbf{x}^*}.$$

The stability of the equilibrium point $\mathbf{x}^*$ depends on the eigenvalues of J. If all eigenvalues have negative real parts, $\mathbf{x}^*$ is asymptotically stable. If at least one eigenvalue has a positive real part, $\mathbf{x}^*$ is unstable. Finally, if all eigenvalues have non-positive real parts, with at least one having zero real part, the linearization test is inconclusive.

Lyapunov's linearization method provides a powerful first step in determining stability, but it has limitations. In cases where eigenvalues lie on the imaginary axis, nonlinear terms may influence stability, requiring a more detailed analysis using Lyapunov functions.

3.1.2 Lyapunov Functions

Lyapunov's second method provides a useful approach for determining the stability of an equilibrium point without explicitly solving the system's differential equations. Instead, stability is inferred by constructing a Lyapunov function, a scalar function that behaves analogously to energy in a physical system. If such a function can be found, it provides a direct means of assessing stability by analyzing its behavior over time.

Consider a system governed by the differential equation

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}),$$

where $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{f}(\mathbf{x})$ is a smooth vector field. A Lyapunov function is a continuously differentiable scalar function $V(\mathbf{x})$ that satisfies certain conditions:

1. $V(\mathbf{x})$ is positive definite: $V(\mathbf{x}^*) = 0$ and $V(\mathbf{x}) > 0$ for $\mathbf{x} \neq \mathbf{x}^*$,

2. The time derivative of $V(\mathbf{x})$ along system trajectories is non-positive:

$$\dot{V}(\mathbf{x}) = \nabla V \cdot \mathbf{f}(\mathbf{x}) \leq 0.$$

   This ensures that $V(\mathbf{x})$ does not increase over time.

These conditions imply that as time progresses, the function $V(\mathbf{x})$ either remains constant or decreases, preventing trajectories from diverging away from the equilibrium point. If $\dot{V}(\mathbf{x}) < 0$ for all $\mathbf{x} \neq \mathbf{x}^*$, then the equilibrium point is asymptotically stable, meaning trajectories approach $\mathbf{x}^*$ over time. If $\dot{V}(\mathbf{x}) \leq 0$ but not necessarily negative definite, the equilibrium point is Lyapunov stable, though trajectories may not necessarily converge to $\mathbf{x}^*$.

3.1.3 Applications of Lyapunov Functions

Lyapunov functions play a central role in engineering applications where stability is a primary concern. They are used extensively in both aerospace engineering and robotics.

In aerospace applications, Lyapunov functions are used to analyze the stability of spacecraft orbital motion and autonomous landing systems. In the determination and control of spacecraft orbits, Lyapunov functions help assess whether small perturbations will cause a spacecraft to drift away from its intended trajectory. Similarly, for spacecraft reentry procedures, Lyapunov stability analysis is used to ensure that corrective maneuvers will keep the reentering craft on a safe path.

In robotics, Lyapunov functions are used in robotic motion control, particularly to ensure the stability of adaptive control systems. Lyapunov-based controllers guarantee stable trajectory tracking in robotic manipulators, allowing for example, robotic arms to reach and maintain a desired position. In autonomous vehicles, Lyapunov functions are used to ensure a robot remains on a desired trajectory despite uncertainties in external disturbances.

3.2 Lyapunov Exponents and Chaos Theory

Although chaotic systems are deterministic, they exhibit behavior that appears random and unpredictable. These systems are highly sensitive to initial conditions, a phenomenon often referred to as the Butterfly Effect. A small perturbation to an initial state can lead to vastly different outcomes, and this is due to nonlinear interactions within the system that amplify small variations over time.

Chaotic systems arise in the form of many real-world phenomena, concerning fields from meteorology to economics. A well-known example of a chaotic system is the weather, which makes long-term weather forecasting inherently unreliable beyond a certain point. The Lorenz attractor, a classic model of atmospheric convection, is the mathematical interpretation of this system. Other chaotic systems include population dynamics in ecology and financial markets in economics.

Lyapunov exponents provide a quantitative measure of sensitivity to initial conditions in dynamical systems. They describe the rate at which two initially close trajectories in the system diverge or converge over time. Mathematically, the Lyapunov exponent λ is defined as:

$$\lambda = \lim_{t \to \infty} \frac{1}{t} \ln \left(\frac{|\delta x(t)|}{|\delta x(0)|} \right),$$

where $|\delta x(t)|$ is the separation between two trajectories at time t, and $|\delta x(0)|$ is the initial separation between them. If $\lambda > 0$ small perturbations grow exponentially over time, indicating chaotic behavior. Conversely, if $\lambda < 0$ perturbations decay, suggesting stable behavior, and if $\lambda = 0$ the system exhibits neutral behavior, such as in systems with periodic orbits.

4 Numerical Investigations of Systems Using the Lyapunov Exponent

4.1 The Double Pendulum

The double pendulum is a well-known example of a chaotic system due to its extreme sensitivity to initial conditions. In this section, we describe the numerical integration of a double pendulum system using a leapfrog integration scheme to examine this sensitivity. This system is governed by the following equations:

$$\ddot{\theta}_1 = \frac{m_2 g \sin\theta_2 \cos(\theta_1-\theta_2) - m_2 \sin(\theta_1-\theta_2)[l_1\dot{\theta}_1^2\cos(\theta_1-\theta_2) + l_2\dot{\theta}_2^2] - (m_1+m_2)g\sin\theta_1}{l_1[m_1 + m_2\sin^2(\theta_1-\theta_2)]}$$

$$\ddot{\theta}_2 = \frac{(m_1+m_2)[l_1\dot{\theta}_1^2\sin(\theta_1-\theta_2) - g\sin\theta_2 + g\sin\theta_1\cos(\theta_1-\theta_2)] + m_2 l_2\dot{\theta}_2^2\sin(\theta_1-\theta_2)\cos(\theta_1-\theta_2)}{l_2[m_1 + m_2\sin^2(\theta_1-\theta_2)]},$$

where a subscript 1 indicates the pendulum attached to the pivot, a subscript 2 indicates the second pendulum, m is the mass of the bob, l is the length of the rod, and g is the gravitational acceleration near the surface of the Earth.

Our system consists of two pendulum rods, each with a length of $l_1 = l_2 = 1$ m and masses $m_1 = m_2 = 1$ kg. The simulation is run for a total time of $T = 50$ with a timestep of $dt = 0.0001$.

For the unperturbed system, the initial angles are set to $\theta_1 = \theta_2 = \pi/2$ radians. To examine the impact of slight changes in initial conditions, the second pendulum's angles are perturbed by 0.001 radians. The resulting trajectories of the pendulum masses, along with the logarithmic separation distance between the two systems, are shown in the figure below.

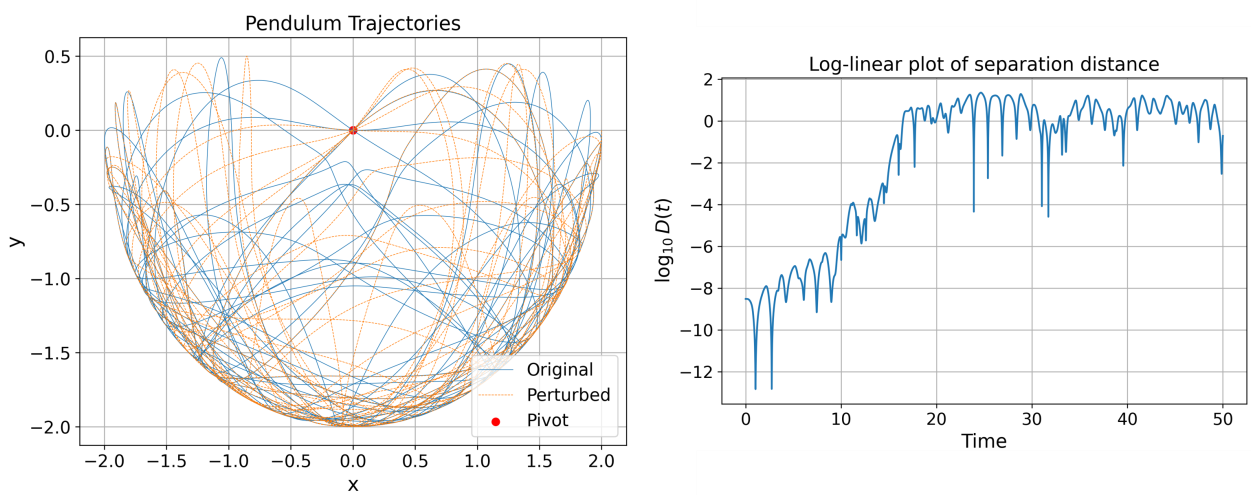


Figure 1: Left: Trajectory of the double pendulum system, with the perturbed system shown in orange. Right: The logarithm of the separation distance between the two systems over time.

The log-linear plot of the separation distance exhibits a characteristic linear increase between $T = 0$ and $T = 20$, indicative of exponential divergence. Beyond this interval, the plot plateaus due to the physical constraint imposed by the fixed rod lengths, which limit the maximum possible separation of the masses. By analyzing data within the time interval $T = 5$ to $T = 15$, we estimate the Lyapunov exponent to be $\lambda = 0.49$, reflecting the chaotic dynamics of the system.

4.2 Three-Dimensional Hill's Equations

4.2.1 Investigation of the L2 Lagrange Point

Hill's equations describe the motion of an object of negligible mass under the influence of the gravitational forces from two primary bodies, a system known as the restricted 3-body problem. These equations are especially useful for studying the dynamics of Lagrange points in a simplified Solar System; for example, a system comprised of the Sun and the Earth. The equations of motion are

$$\frac{d^2x}{dt^2} = -\frac{3x}{r^3} + 2\frac{dy}{dt} + 3x$$
$$\frac{d^2y}{dt^2} = -\frac{3y}{r^3} - 2\frac{dx}{dt}$$
$$\frac{d^2z}{dt^2} = -\frac{3z}{r^3} - z.$$

We examine the behavior of an orbit starting near the L2 Lagrange point, with initial coordinates $(x_0, y_0, z_0) = (0.9999, 0.00005, 0.00005)$. A second particle is introduced with an initial position perturbed by $\epsilon = 10^{-6}$ in x_0, y_0, and z_0. The system is integrated using a fourth-order Runge-Kutta scheme up to $T = 10$ with a timestep of $dt = 0.0001$.

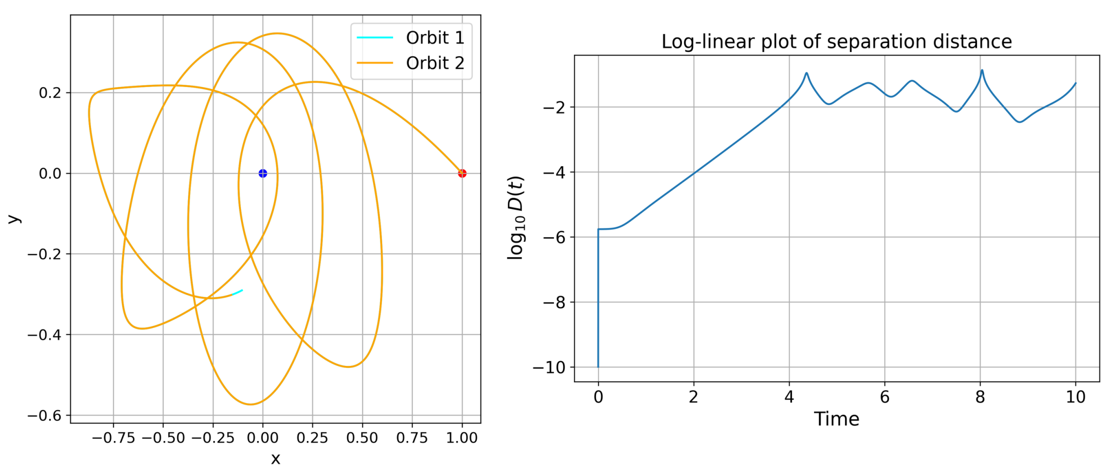


Figure 2: Left: Trajectories of particle 1 (cyan) and perturbed particle 2 (orange) in the xy-plane. Right: Logarithm of separation distance between particle 1 and particle 2 over time.

Figure 2 shows the logarithm of the separation distance between particle 1 and particle 2. After an initial jump, the plot exhibits sharp oscillations that appear somewhat periodic, oscillating around a horizontal line. However, this data is insufficient to reasonably approximate the Lyapunov exponent. To investigate further, we extended the total run time to $T = 50$ with the same initial conditions. Figure 3 shows the resulting trajectories and log-linear distance plots.

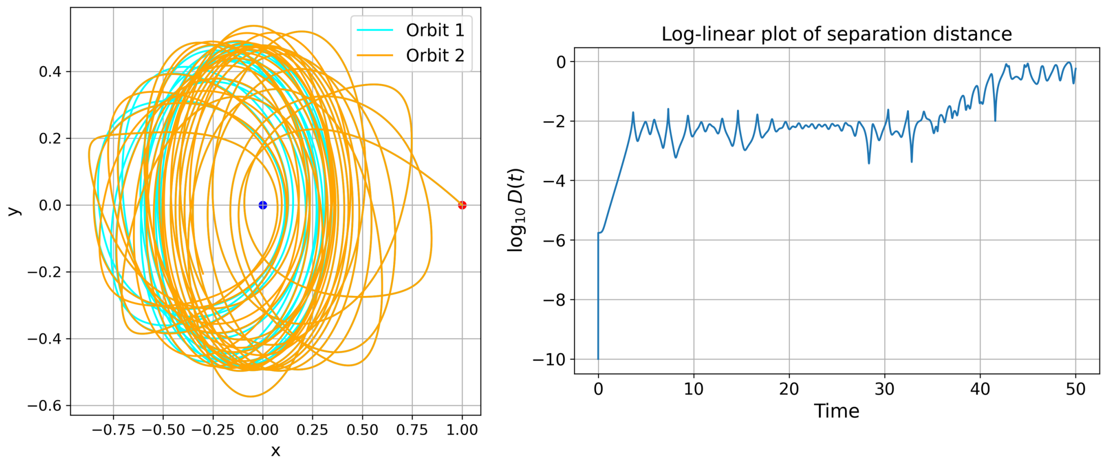
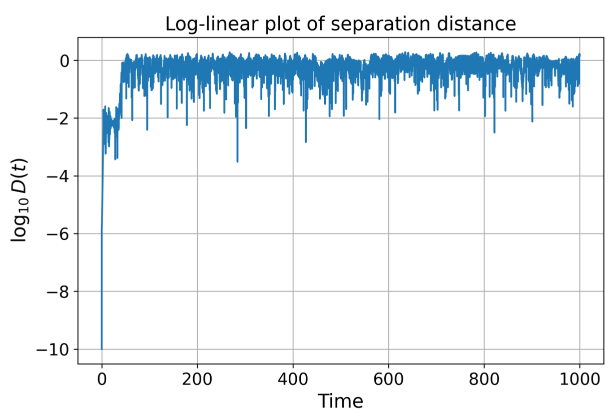


Figure 3: Left: Trajectories of particle 1 (cyan) and perturbed particle 2 (orange) in the xy-plane. Right: Logarithm of separation distance between particle 1 and particle 2 over time.

In Figure 3, the trajectories of particle 1 and particle 2 no longer overlap. The logarithm of the separation distance oscillates around a horizontal line until roughly $T = 30$, after which a linear trend emerges until approximately $T = 40$. During this interval, we approximate a Lyapunov exponent of $\lambda = 0.53$. Qualitatively, the system appears neutrally stable for an extended period before undergoing instability that separates the trajectories. Remarkably, this neutral stability persists over long timescales, beyond reasonable computational limits in Python. This is illustrated in the figure below, where the total integration time is increased to $T = 1000$.



4.2.2 Varying the Starting Position

Another line of investigation involved varying the starting coordinate to examine its effect on system stability. Bringing the starting position closer to the planet resulted in interesting orbital patterns. These trajectories also appeared neutrally stable, exhibiting similar oscillations in the log-linear separation distance plots. Some examples are shown below.

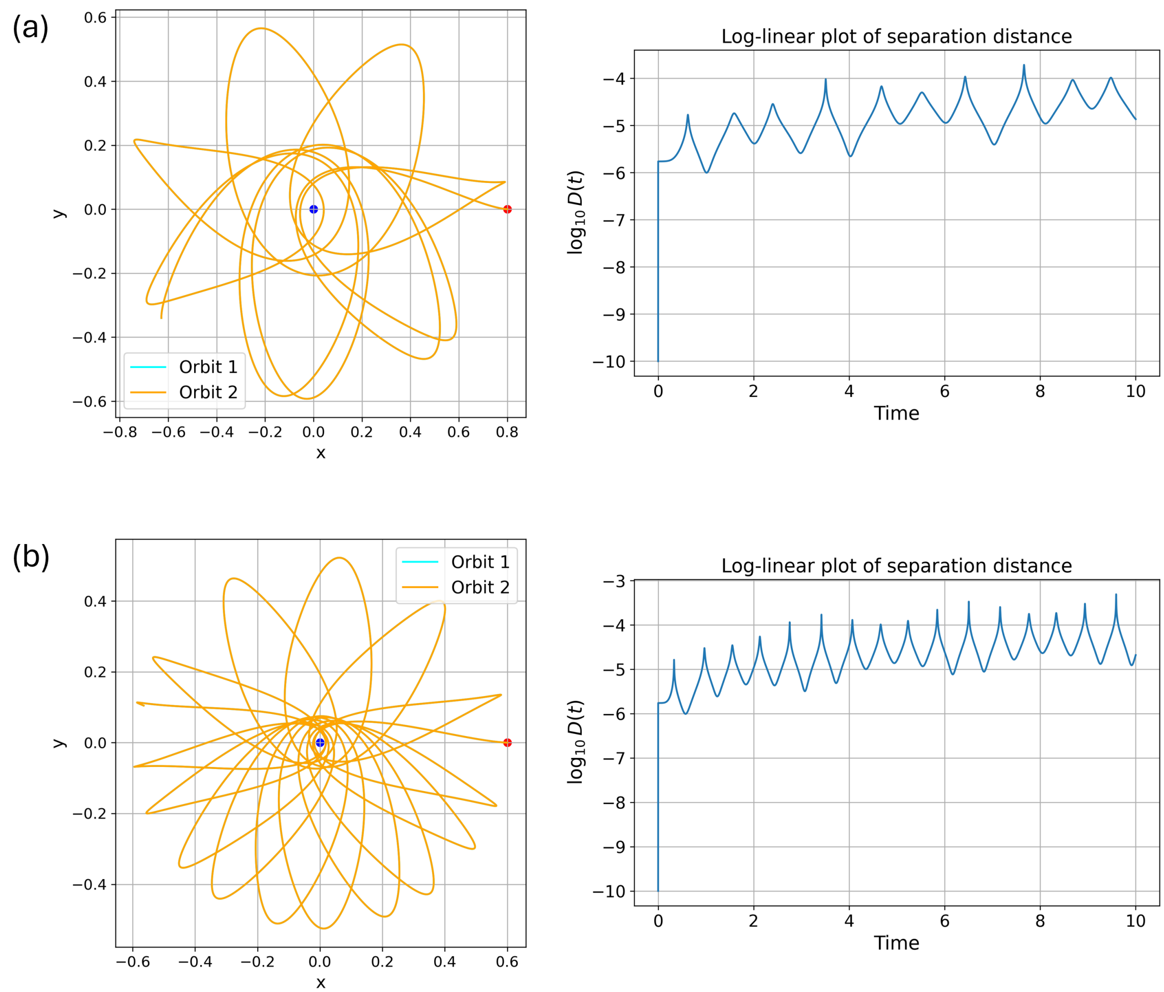


Figure 4: (a) Starting position $(x_0, y_0, z_0) = (0.7999, 0.0005, 0.0005)$ and (b) starting position $(x_0, y_0, z_0) = (0.5999, 0.0005, 0.0005)$. Trajectories are shown on the left, with log-linear plots of separation distance on the right.

As x_0 decreases, the number of orbits increases, causing the log-linear plot of separation distance to oscillate at a higher frequency. At roughly $x_0 = 0.3999$, the particles spiral into the planet and appear to be ejected, leading to trajectory divergence.

4.2.3 Future Improvements

Several avenues for future improvement could enhance the understanding of stability near the L2 Lagrange point. First, extending the total integration time beyond $T = 1000$ could provide deeper insights into long-term stability and the persistence of neutral behavior. Increasing computational efficiency, potentially through more optimized numerical methods such as symplectic integrators, would make these longer simulations more feasible.

Another potential improvement is to explore a wider range of initial conditions, particularly with finer perturbations in x_0, y_0, and z_0. This could help determine how sensitive the Lyapunov exponent is to small changes and whether certain initial conditions are more prone to rapid divergence. Additionally, systematically varying parameters such as mass ratio and rotation rate could reveal how these factors influence stability near the L2 point.

5 Conclusion

Aleksandr Mikhailovich Lyapunov's profound influence on mathematics and its applications is evident through his groundbreaking work in stability theory, probability theory, and nonlinear dynamics. Lyapunov's development of stability theory, particularly through his linearization method and Lyapunov functions, provided a framework for determining the stability of dynamical systems. The introduction of Lyapunov exponents offered a quantitative measure for the sensitivity to initial conditions in chaotic systems, which has become essential in understanding complex behavior across fields such as meteorology, economics, and celestial mechanics.

Our numerical investigations demonstrated the practical significance of Lyapunov exponents in analyzing chaotic dynamics. In the double pendulum system, we observed exponential divergence with a Lyapunov exponent of $\lambda = 0.49$, reflecting its well-known sensitivity to initial conditions. Similarly, for three-dimensional Hill's equations near the L2 Lagrange point, we identified periods of neutral stability followed by instability with an estimated Lyapunov exponent of $\lambda = 0.53$. Varying initial conditions revealed intriguing orbital patterns, with log-linear plots exhibiting oscillations that increased in frequency as the starting position approached the planet.

Future work could benefit from extended integration times and more efficient numerical methods to better capture long-term stability. Exploring a broader range of initial conditions and systematically varying parameters such as mass ratio and rotation rate could provide deeper insights into the sensitivity of the Lyapunov exponent and the conditions that lead to rapid divergence.

References

[1] Steven H. Strogatz. *Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering*. 3rd. Westview Press, 2018. ISBN: 978-0813349107.

[2] László Hatvani. "Aleksandr Lyapunov, the man who created the modern theory of stability". In: *Electronic Journal of Qualitative Theory of Differential Equations* 26 (2019), pp. 1–9. DOI: 10.14232/ejqtde.2019.1.26.

[3] Ege Cagri Altunkaya et al. "Stability and Safety Assurance of an Aircraft: A Practical Application of Control Lyapunov and Barrier Functions". In: *SSRN Electronic Journal* (2023). DOI: 10.2139/ssrn.4823223. URL: http://dx.doi.org/10.2139/ssrn.4823223.

[4] Victor Santibáñez and Rafael Kelly. "Strict Lyapunov functions for control of robot manipulators". In: *Automatica* 33.4 (1997), pp. 675–682. ISSN: 0005-1098. DOI: https://doi.org/10.1016/S0005-1098(96)00194-X.