PHYD38 Soliton

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1 Introduction

Solitons are solitary waves that propagate with constant velocity while maintaining their shape. A soliton is a fundamental and widely occurring type of nonlinear wave. In contrast to typical linear wave-trains, a soliton is highly localized and characterized by a single, well-defined peak—hence the term "solitary wave." Remarkably, it maintains its shape during propagation and even after interacting with other solitons. This stability arises from a delicate balance between two competing effects: nonlinearity, which causes higher amplitude waves to travel faster, and dispersion, which tends to spread waves of different frequencies at varying speeds.[1]

In the nineteenth century, solitons were first observed by the Scottish civil engineer John Scott Russell. Russell observed that when a canal barge struck an underwater obstruction and stopped suddenly, the bow wave did not dissolve into numerous little ripples through dispersion. Instead, a smooth, bell-shaped crest, approximately a half meter high and independent of the cross-channel direction, emerged from the forth. He followed the stable and unchanging crest for several kilometers until it disappeared.[2]

In this paper, we will discuss the derive the solution of the solitons, the stability of the k-dV equation by using the xxxx method and calculate solitons' propagation and interaction.

2 Theory

2.1 Characteristics of Russell's Solitary Wave

This type of wave later termed the "Russell's solitary wave" or "soliton". The solitary wave has the following characteristics:

1. Permanent form: The wave is long and shallow, with amplitude a much smaller than the wavelength

$$(\lambda): \quad \frac{a}{\lambda} \ll 1.$$

2. The speed of the solitary wave is given by:

$$c^2 = g(h + a) \tag{1}$$

where:

- c is the wave speed,
- g is the gravitational acceleration,
- h is the constant depth of the channel,
- a is the wave amplitude.

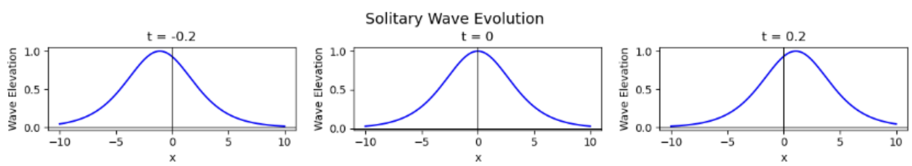

Figure 1: Solitary wave evolution at t= -0.2 , t=0 and t0.2

Figure 1 shows a solitary wave subject to gravitational acceleration g=9.8 m/s in a channel of uniform depth h=2.0 with amplitude a=1.0. Substituting the given parameters yields $c = \sqrt{29.4}$.

At this time, Russell's observations conflicted with Airy's linear wave theory (1841), which predicted that small amplitude waves could not maintain a constant profile in finite-depth water. This contradiction led to the development of more advanced nonlinear wave theories.[2]

2.2 Contributions of Boussinesq and Lord Rayleigh

In 1871, Boussinesq and, in 1876, Lord Rayleigh provided theoretical explanations for Russell's solitary wave. Both based their analyses on the equations of motion for an ideal fluid, assuming it to be incompressible and inviscid. They considered the solitary wave to have a wavelength λ_0 much larger than the water depth h, specifically:

$$\delta^2 = \left(\frac{h}{\lambda_0}\right)^2 \ll 1 \tag{2}$$

where δ^2 is called the square of the frequency dispersion parameter.

2.2.1 Boussinesq(Dynamic)

Boussinesq indicated that Airy neglected the vertical acceleration in his wave theory which is responsible for dispersion. Boussinesq derived the wave profile as:

Governing Equation

Boussinesq starts from the shallow water wave equation, modified to include both nonlinear and dispersion effects:

$$\frac{\partial^2 \eta}{\partial t^2} - gh\frac{\partial^2 \eta}{\partial x^2} = gh\frac{\partial^2}{\partial x^2}\left(\frac{3}{2h}\eta^2 + \frac{h^2}{3}\frac{\partial^2 \eta}{\partial x^2}\right) \tag{3}$$

where $\eta(x,t)$ represents the free surface elevation.

Traveling Wave Ansatz

Assume a traveling wave solution:

$$\eta(x,t) = \eta(\xi), \quad \xi = x - ct \tag{4}$$

Then, time derivatives become:

$$\frac{\partial}{\partial t} = -c\frac{d}{d\xi} \tag{5}$$

Substitute into the equation and integrate appropriately, simplifying to obtain an ordinary differential equation in ξ.

Solitary Wave Solution

After simplification, Boussinesq derives the solution:

$$\eta(x,t) = a\,\mathrm{sech}^2\left(\beta(x - ct)\right) \tag{6}$$

for any $a > 0$. is the amplitude and:

$$\beta^2 = \frac{3a}{4h^2(h+a)}$$

resulting in a stable solitary wave profile.

- a: Amplitude, wave height,
- β: Controls the width of the wave,
- c: Wave speed,
- λ_0: Wavelength related to β,
- The specific expression of β^2 indicates that the larger the amplitude, the narrower the wave.

Figure 2: Enter Caption

The waveform is a function sech^2 that is sharply peaked and decays rapidly to zero at infinity, ensuring the wave is localized and does not disperse. (like an isolated crest), this is correct only if (2) is satisfied, which means in the shallow water.

2.2.2 Rayleigh(Static)

Lord Rayleigh treated the problem as a steady motion (time-independent), and formulated the following ODE:

$$\left(\frac{dy}{dx}\right)^2 = \frac{3(y-h)^2}{h^2}\left(1-\frac{gy}{c^2}\right) \tag{7}$$

The ODE (4) governs long one-dimensional, small amplitude surface gravity waves in a channel of water with uniform depth h, where c represents the uniform velocity of the fluid far from the wave, both ahead and behind.

Detailed Derivation of Rayleigh's ODE Solution

Starting with the ordinary differential equation:

$$\left(\frac{dy}{dx}\right)^2 = \frac{3(y-h)^2}{h^2}\left(1-\frac{gy}{c^2}\right) \tag{8}$$

Step 1: Take the square root

$$\frac{dy}{dx} = \pm\frac{\sqrt{3}(y-h)}{h}\sqrt{1-\frac{gy}{c^2}} \tag{9}$$

Step 2: Variable separation

$$\frac{dy}{(y-h)\sqrt{1-\frac{gy}{c^2}}} = \pm\frac{\sqrt{3}}{h}dx \tag{10}$$

Step 3: Substitution

Let:

$$Y = y-h \quad \Rightarrow \quad y = Y + h$$

Then:

$$\frac{dY}{Y\sqrt{1-\frac{g(Y+h)}{c^2}}} = \pm\frac{\sqrt{3}}{h}dx \tag{11}$$

Step 4: Simplify the square root

Using:

$$c^2 = g(h+a)$$

We get:

$$1 - \frac{g(Y+h)}{g(h+a)} = \frac{a-Y}{h+a} \tag{12}$$

Thus:

$$\frac{dY}{Y\sqrt{\frac{a-Y}{h+a}}} = \pm\frac{\sqrt{3}}{h}dx \tag{13}$$

Step 5: Simplify

$$\frac{dY}{Y\sqrt{a-Y}} = \pm\frac{\sqrt{3(h+a)}}{h}dx \tag{14}$$

Step 6: Integrate

Left side:

$$\int \frac{dY}{Y\sqrt{a-Y}} = -\frac{2}{\sqrt{a}}\tanh^{-1}\left(\frac{\sqrt{a-Y}}{\sqrt{a}}\right) + C_1 \tag{15}$$

Right side:

$$\pm\frac{\sqrt{3(h+a)}}{h}x + C_2$$

Step 7: Combine constants

$$-\frac{2}{\sqrt{a}}\tanh^{-1}\left(\frac{\sqrt{a-Y}}{\sqrt{a}}\right) = \pm\frac{\sqrt{3(h+a)}}{h}x + C \tag{16}$$

Step 8: Solve for $y(x)$

After simplification:

$$\tanh^{-1}\left(\frac{\sqrt{a-Y}}{\sqrt{a}}\right) = -\frac{\sqrt{a}}{2}\left(\pm\frac{\sqrt{3(h+a)}}{h}x + C\right) \tag{17}$$

Using hyperbolic function identities, final solution:

$$y(x) = h + a\,\mathrm{sech}^2(\beta(x-x_0)) \tag{18}$$

where:

$$\beta = \sqrt{\frac{3a}{4h^2(h+a)}}$$

The constant x_0 is determined by the integration constant C and represents the horizontal shift of the wave profile.

From ODE (4), it can be seen that the elevations where the tangents to the solution curves are horizontal occur when $y = h$ or when $y = \frac{c^2}{g}$. Since $1 - \frac{gy}{c^2}$ is non-negative, the maximum elevation corresponds to:

$$y_{\max} = \frac{c^2}{g} \tag{19}$$

This implies there is no depression below the free surface in the wave profile, meaning the wave has only one elevation. Denoting the maximum height above the free surface by a, we obtain:

$$c^2 = gy_{\max} = g(h+a) \tag{20}$$

Solving leads to the solitary wave speed:

$$c^2 = g(h+a) \tag{21}$$

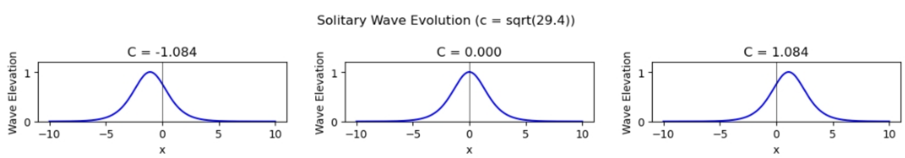

Figure 3: Russell's solitary wave as explained by Lord Rayleigh for $c = \sqrt{29.4}$, g = 9.8, and h = 2.

Figure 2 presents the same solitary wave profile as shown in Figure 1. Lord Rayleigh's solution incorporates the channel depth h into the formulation and focuses on a steady-state scenario, without considering time dependence. In contrast, Boussinesq introduced time as a variable to capture the dynamic evolution of the wave, making his approach suitable for explaining the propagation behavior of solitary waves.

The Ursell number, denoted as U, is introduced to characterize the balance between nonlinearity and dispersion in long surface gravity waves. It is defined as:

$$U = \frac{a\lambda_0^2}{h^3} = \frac{a/h}{\delta^2} \tag{22}$$

where:

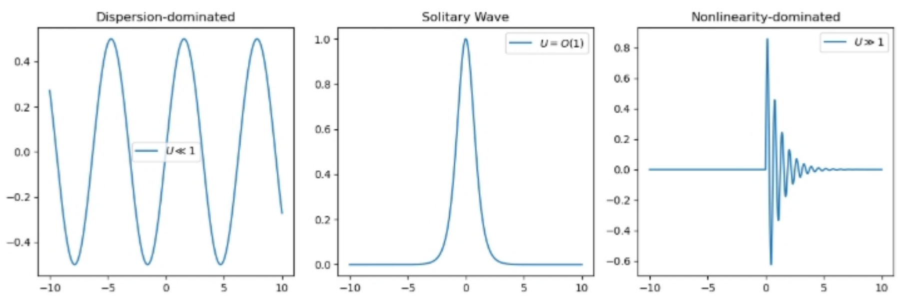

Figure 4: Ursell numers

- a = amplitude (wave height),
- h = water depth,
- λ_0 = wavelength,
- $\delta^2 = \left(\frac{h}{\lambda_0}\right)^2$ = dispersion parameter.

Physical Meaning

The Ursell number U measures the relative strength of the nonlinearity in the wave. Specifically:

- When $U \ll 1$, dispersion dominates, and the wave behaves almost linearly,
- When $U = O(1)$, nonlinearity and dispersion are balanced, leading to the formation of solitary waves,
- When $U \gg 1$, nonlinearity dominates, and strong wave steepening or breaking may occur.

Figure 3 illustrates that when $U = O(1)$, a balance is achieved between nonlinearity and dispersion. Nonlinearity acts to steepen and amplify the wave crest, whereas dispersion tends to spread and flatten the wave profile. This interplay results in the formation of a solitary wave—a localized, stable, single-peaked structure that can propagate over long distances while maintaining its shape.

3 Soliton Perturbation

This part uses the energy method to prove the linear stability of KDV solitons under small perturbations. By introducing a small perturbation and linearizing the system, we derive an energy functional that quantifies the deviation from the soliton profile. Proving that the energy functional does not increase with time, we ensure the linear stability of the soliton against perturbations.

Introducing a small perturbation ε:

$$u(x,t) = u(x,t) + \varepsilon(x,t) \tag{23}$$

Substituting the new solution into the KDV function gives the following.

$$(u+\varepsilon)_t - 6(u+\varepsilon)(u+\varepsilon)_x + (u+\varepsilon)_{xxx} = 0 \tag{24}$$

Expanding the equation:

$$u_t - 6uu_x + u_{xxx} + \varepsilon_t - 6\varepsilon u_x - 6u\varepsilon_x - 6\varepsilon_x\varepsilon + \varepsilon_{xxx} = 0 \tag{25}$$

Since u is a solution of the KDV equation, after ignoring higher order terms, the equation simplifies to:

$$\varepsilon_t - 6(u\varepsilon)_x + \varepsilon_{xxx} = 0 \tag{26}$$

Define the energy functional as:

$$E(t) = \frac{1}{2}\int_{-\infty}^{\infty} \varepsilon_x^2 \, dx \tag{27}$$

We chose ε_x instead of ε because the soliton solution is in the form x-ct, which the soliton will retain its form under spatial translations. This will introduce a zero mode in the linearized perturbation, it is the perturbation in the form: $\varepsilon_{\text{zero}}(x,t) \propto \partial_x u(x-ct)$ this kind of perturbation will only shift the soliton position without altering its stability or shape. It does not act like a perturbation but will still increase the functional energy if we choose ε.

The time derivative of the energy fractional is:

$$\frac{dE}{dt} = \int_{-\infty}^{\infty} \varepsilon_x \varepsilon_{xt} \, dx = \int \varepsilon_x \partial_x \left(-\varepsilon_{xxx} + 6(u\varepsilon)_x\right) dx \tag{28}$$

The equation has four individual terms:

$$\int_{-\infty}^{\infty} (-\varepsilon_x \varepsilon_{xxxx}) + (6u_x x \varepsilon_x \varepsilon) + (12u_x \varepsilon_x^2) + (6u\varepsilon_x\varepsilon_{xx}) \, dx \tag{29}$$

For the first term, we use integration by parts:

$$\int_{-\infty}^{\infty} -\varepsilon_x \varepsilon_{xxxx} \, dx = -\varepsilon_x \varepsilon_{xxx}\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \varepsilon_{xxx}\varepsilon_{xx} \, dx = 0 + \frac{1}{2}\varepsilon_{xx}^2\Big|_{-\infty}^{\infty} = 0 \tag{30}$$

This is because the KDV equation obeys the conservation of (horizontal momentum):

$$\int_{-\infty}^{\infty} u^2 \, dx = constant \tag{31}$$

This conservation law bounds the evolution of perturbation, it requires the perturbation to be square integrable, meaning it must be 0 at infinite.

Before we integrate the other parts of the equation. We need to look at some properties of the soliton solution, it has the form of $\text{sech}^2(x)$. This is an even function. And its first, second, and third derivatives are odd, even, and odd function, respectively.

Now for the second term:

$$\int_{-\infty}^{\infty} 6u_{xx}\varepsilon_x\varepsilon \, dx = \int_{-\infty}^{\infty} 3u_{xx}(\varepsilon^2)_x \, dx = -3\int_{-\infty}^{\infty} u_{xxx}\varepsilon^2 \, dx \tag{32}$$

Since the third derivative of u is an odd function, and the integration is from negative to positive infinity, the integration is zero.

For the third term:

$$\int 12u_x\varepsilon_x^2 \, dx = 0 \tag{33}$$

Since it is an odd function,

The last term:

$$\int_{-\infty}^{\infty} 6u\varepsilon_x\varepsilon_{xx} \, dx = \int_{-\infty}^{\infty} 3u(\varepsilon_x^2)_x \, dx = -3\int_{-\infty}^{\infty} u_x\varepsilon_x^2 \, dx = 0 \tag{34}$$

Since the first derivative of u is also an odd function.

Thus, we have proven that the energy does not increase with time, meaning that the perturbation is linearly stable.

4 Korteweg-de Vries (KdV) Equation

The Korteweg-de Vries (K-dV) equation is a partial differential equation that describes certain types of wave phenomena. It is given by:

$$u_t - 6uu_x + u_{xxx} = 0 \tag{35}$$

This was independently dervied by Boussinesq in 1877 and later by Diederik Korteweg and Gustav de Vrise in 1895. It plays a significant role in modeling wave propagation in shallow water environments. [5]

A Brief Derivation of the Solution

For the single soliton solution, let $u(x,t) = f(X) = f(x - ct)$, where c is the wave speed. Substituting into the KdV equation:

$$
\begin{aligned}
0 &= -cf' - 6ff' + f''' \\
f''' &= cf' + 6ff' = cf' + 3(f^2)' \qquad (36) \\
f'' &= cf + 3f^2 + A \\
(f')^2 &= cff' + 3f^2 f' + Af' \\
\frac{1}{2}\left[(f')^2\right]' &= \frac{c}{2}(f^2)' + (f^3)' + Af' \qquad (37)
\end{aligned}
$$

Since solitons are localized $f, f', f'' \to 0$ as $X \to \pm\infty$, we set $A = 0$, yielding:

$$
\frac{1}{2}(f')^2 = f^3 + \frac{c}{2}f^2 = f^2\left(f + \frac{c}{2}\right) \qquad (38)
$$

One can perform a transformation and obtain:

$$
f' = \sqrt{2f^2\left(f + \frac{c}{2}\right)} \qquad (39)
$$

Then, by changing variables on both sides, we can write:

$$
\frac{df}{\sqrt{2f^2\left(f + \frac{c}{2}\right)}} = dX \qquad (40)
$$

This integral is tractable, and the solution is:

$$
f(X) = -\frac{c}{2}\operatorname{sech}^2(\theta) = -\frac{c}{2}\operatorname{sech}^2\left(\frac{\sqrt{c}}{2}(X - X_0)\right) \qquad (41)
$$

As for the multi-soliton solutions of the KdV equation, since the equation is integrable[1], analytical solutions can be obtained using the Hirota method. The original function $u(x,t)$ can be expressed in terms of a τ-function as:

$$
u(x,t) = 2\frac{\partial^2}{\partial x^2}\left[\ln F(x,t)\right], \qquad (42)
$$

where

$$
F(x,t) = 1 + \sum_{i=1}^{N} e^{\theta_i} + \sum_{1 \le i < j \le N} A_{ij} e^{\theta_i + \theta_j} + \cdots \text{ (a sum of multiple exponential terms)},
$$

and

$$
\theta_i = k_i x - 4k_i^3 t + \theta_{i0}. \qquad (43)
$$

For the KdV equation, there is a well-known result:

[1] 1

$$A_{ij} = \left(\frac{k_i - k_j}{k_i + k_j}\right)^2. \tag{44}$$

For the two-soliton case (i.e., $N = 2$), the function F can be simplified as:

$$F(x,t) = 1 + e^{\theta_1} + e^{\theta_2} + A_{12}e^{\theta_1+\theta_2}, \tag{45}$$

where

$$\theta_1 = k_1x - 4k_1^3t + \theta_{10}, \quad \theta_2 = k_2x - 4k_2^3t + \theta_{20},$$

and

$$A_{12} = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2.$$

Therefore, the two-soliton solution can be written as:

$$u(x,t) = 2\frac{\partial^2}{\partial x^2}\ln\left(1 + e^{\theta_1} + e^{\theta_2} + \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 e^{\theta_1+\theta_2}\right).$$

$$= 2\,\frac{\partial^2}{\partial x^2}\ln\left[1 + e^{k_1x-4k_1^3t+\theta_{10}} + e^{k_2x-4k_2^3t+\theta_{20}} + \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 e^{(k_1+k_2)x-4(k_1^3+k_2^3)t+\theta_{10}+\theta_{20}}\right] \tag{46}$$

Here, θ_{10} and θ_{20} represent the initial positions of the two solitons.

Consistency between the Hirota method and the single-soliton ODE solution

When it degenerates into a single soliton, we have $u(x,t) = 2\frac{\partial^2}{\partial x^2}\ln F(x,t)$ with $F = 1 + e^{\theta_1}$.

Let:

$$\theta_1 = kx - 4k^3t + \theta_0,$$

then:

$$u(x,t) = 2\frac{\partial^2}{\partial x^2}\ln\left(1 + e^{kx-4k^3t+\theta_0}\right) = \frac{k^2}{2}\,\text{sech}^2\left(\frac{1}{2}(kx - 4k^3t + \theta_0)\right). \tag{47}$$

Consistent with the form of the single-soliton solution above.

5 Numerically computation

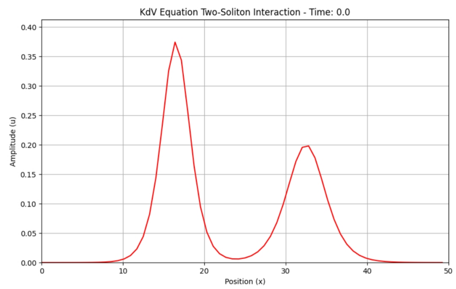

Figure 5: At t=0

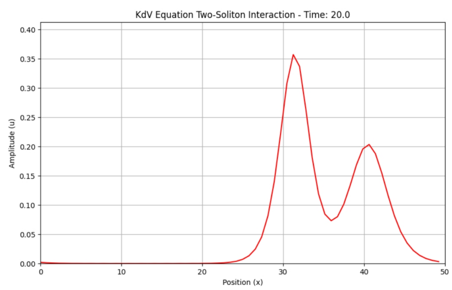

Figure 6: At t=20

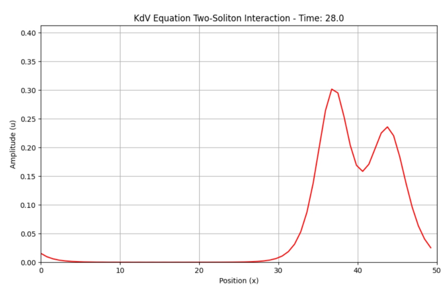

Figure 7: At t=28

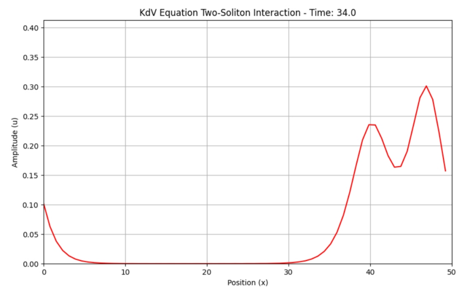

Figure 8: At t=34

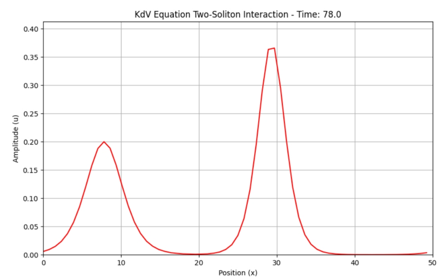

Figure 9: At t=78

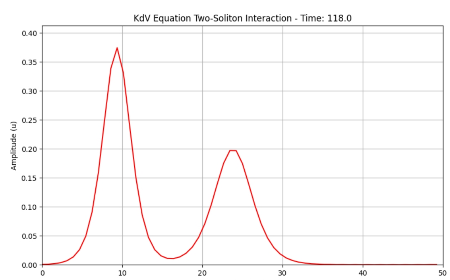

Figure 10: At t=118

These diagrams illustrate the typical interaction behavior of two solitons during their propagation. Initially, the two solitons are well-separated, each maintaining a stable shape. As time progresses, the faster soliton gradually approaches and collides with the slower one. During the collision, their waveforms exhibit nonlinear distortion rather than simple linear superposition. After the interaction, both solitons recover their original shape and speed, continuing to propagate independently with only a slight phase shift in position. This process highlights the soliton's remarkable stability and resilience, which is a defining feature of soliton interactions described by the KdV equation.[10]

6 Conclusion

This study examined the key properties of Russell's solitary wave, characterized by its permanent, localized form and speed relation $c^2 = g(h + a)$. Boussinesq's dynamic approach incorporated nonlinearity and dispersion, yielding a stable sech^2 wave solution. Rayleigh's static method led to the same result through a steady-state analysis. The Ursell number was introduced to describe the balance between nonlinearity and dispersion, highlighting conditions under which solitary waves form. Together, these contributions deepen our understanding of solitary wave behavior in shallow water.

Proof of the integrability of the KdV equation[1]

To determine whether a system is integrable, the most commonly used method is to find a Lax pair such that

$$\frac{dL}{dt} = [P, L],$$

where L and P are operators.

For the KdV equation, let

$$L = -\frac{d^2}{dx^2} + u(x, t), \quad P = -4\frac{d^3}{dx^3} + 6u\frac{d}{dx} + 3u_x.$$

$$\frac{\partial L}{\partial t} = \frac{\partial}{\partial t}\left(-\frac{d^2}{dx^2} + u\right) = \frac{\partial u}{\partial t} = -6uu_x - u_{xxx}$$

$$[P, L] = PL - LP$$

$$PL\psi = \left(-4\partial_x^3 + 6u\partial_x + 3u_x\right)\left(-\psi'' + u\psi\right)$$

$$LP\psi = \left(-\partial_x^2 + u\right)\left(\left(-4\partial_x^3 + 6u\partial_x + 3u_x\right)\psi\right)$$

Here, ψ is an arbitrary function. Then one can combine $PL\psi$ and $LP\psi$,

$$[P, L] = -6uu_x - u_{xxx} = \frac{dL}{dt}$$

Hence the integrability of KdV equation.

After we perform numerical simulations, we observe that solitons temporarily deform during collision due to nonlinear interaction, but after the collision, they retain their original shape and speed, only exhibiting a slight phase shift. This stability is a key feature distinguishing solitons from ordinary waves.

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