Final exam PHYD38: Nonlin. Phys and Chaos, 22 Apr 2025

1 Written part [19% of course mark]

Relative weights for each problem are indicated.

1.1 [20p.] Linear stability of triangular Largrange points

In the circular 2-Body problem, two gravitating bodies, the more massive one with mass $1-\mu$ and the less massive with mass μ, move on circles around a common center of mass.

Giuseppe Luigi Lagrangia, a.k.a. Joseph Louis Lagrange, proved that in Cartesian frame rtating with the binary system, there are 5 fixed points, 3 collinear (about which you know from term work that they are always unstable, being the saddle points of potential U), and two triangular points L4 and L5, forming an ideal equilateral triangle with the two massive bodies.

If we choose such units of distance and time that the angular speed of the binary system, gravitational constant, and the sum of masses are all equal to 1, then the following linearized equations of motion of the restricted 3-Body problem govern the motion of a 3rd particle (test particle of zero mass) in the vicinity of a fixed point:

$$\ddot{x} = -U_{xx}\,x - U_{xy}\,y + 2\dot{y}$$

$$\ddot{y} = -U_{xy}\,x - U_{yy}\,y + 2\dot{x}$$

where U is the effective potential including the gravitational and centrifugal effects, $U_{xx} = \partial^2 U/\partial x^2$, $U_{yy} = \partial^2 U/\partial y^2$, and $U_{xy} = \partial^2 U/(\partial x \partial y)$ is the mixed derivative. Derivatives are evaluated at the fixed point L4 = (0,0), so they are all real constants, while variables x and y are the small deviations from the Lagrange point.

The second derivatives at L4 are: $U_{xx} = -3/4$, $U_{yy} = -9/4$, and $U_{xy} = -\frac{3}{4}\sqrt{3}\,(1-2\mu)$.

Analyze the linear stability of the L4 point (L5 has the same stability). It may seem that, since the effective potential U has a maximum (not a minimum!) at $(0,0)$, as evidenced by negative second derivatives, the L4 point should be unstable. That would certainly be true without frame rotation. But because of significant Coriolis force (terms proportional to $\dot{x}$ and $\dot{y}$), this impression may be wrong. In fact, stability depends on the parameter μ, which is the mass of the smaller body divided by the sum of two attracting masses.

Give both coordinates the standard trial time dependence $e^{\lambda t}$, and form a 2-by-2 matrix eigenproblem. Show that the characteristic equation is biquadratic (quadratic function of λ^2). Derive the eigenvalues λ. No need to compute eigenvectors, stability/instability follows from the eigenvalues.

For which μ is the triangular point stable, and for which unstable? Derive and evaluate the critical bifurcation parameter value μ_s where stability is marginal.

1.2 [20 p.] Nonlinear 2-D system

$$\dot{x} = 4 - y - x^2$$

$$\dot{y} = y\,(x-1)$$

where x and y are real variables. Draw the nullclines, find and characterize all the fixed points. In a separate drawing, sketch the flow in the phase plane.

Is there a closed cycle in your phase portrait? Could there be an unrecognized cycle somewhere? Can a closed orbit cross a nullcline? Use index theory to narrow down the possible locations of a putative closed orbit.

How you would continue, in order to give a definitive answer to the question, if you had more time?

1.3 [18 p.] Discrete, nonlinear mapping

A. Sketch a bifurcation diagram (fixed point values x_* versus parameter β) for the following mapping, in which all values are real numbers:

$$x_{n+1} = \beta\, x_n\, \exp(-x_n)\,.$$

Find all the fixed points and their stability. Draw the $x_*(\beta)$ curves as either solid (stable) or dashed (unstable) branches. Derive the critical values of β where stability begins and ends, and the corresponding range of x_*. Name the bifurcation type you see in the bifurcation diagram.

Pick one value of β for which the mapping should be stable (converging) and illustrate this doing the first 10 iterations on the calculator, starting from $x_0 = 1$. Repeat with another parameter that slightly exceeds the boundary of stability. Record the x_n sequences in your exam booklet. Speculatively sketch the period doubling, i.e. how the period-2 curves might extend your period-1 bifurcation diagram.

HINTS: Use the fixed point condition to simplify expressions that you encounter. It may be easier to sketch $\beta(x_*)$ before sketching $x_*(\beta)$.

1.4 [15p.] Sierpinski triangle and cube

A. Take an equilateral triangular carpet. Remove the inscribed equilateral triangle in the middle, with vertices located at the midpoints of the original triangle's edges. Continue the process ad infinitum. Sierpinski triangle carpet is a self-similar limit of the iteration. Derive the carpet's area and the length of all the borders.

Prove that it is a fractal, by computing the similarity and box dimensions, and showing it is not a whole number.

B. Study in the same manner the Sierpinski cube, in which any full cube (incl. the initial one) is repeatedly subdivided into 27 of 1/3-scale cubes (1/3 edge length), and a number of those cubes are removed to make square tunnels through the cube. Viewed along *any* of the 3 edge directions of the original cube, an empty square tunnel appears in the middle of each face of the cube (cutting out 1/9 of the face). What are the volume, box & similarity dimensions of Sierpinski 1/3-cube? What is the area of this fractal sponge?

2 Quiz [18% of course mark]

Circle Y (yes) or N (no). If you circle N, also circle from one to three words that do are incorrect in that sentence. You earn points only if the right reasons for sentence not being correct are given. Please ignore simple typos and/or grammatical errors.

[N] System $\dot{x} = x - x^2$ has a potential $V = (1/2)\mathbf{x} + (\mathbf{1/3})x^3 + const.$

[Y] There is a theorem for nonlinear centers of conservative 2-D systems: If a conserved quantity E($\mathbf{x}$) exists and has a minimum at an isolated fixed point on a plane $\mathbf{x}$, then all trajectories close to this fixed point are closed.

[Y] Can a 4-dimensional system not have oscillations?

[N] Chaotic oscillations of forced, damped, double pendulum are **intermittently periodic**

[N] Secular term is a **periodic** part of solution (e.g. of resonantly forced oscillator) that grows like time t.

[N] Is the critical slowing down occurring in the system $\dot{x} = rx - x^3$ for $\mathbf{r = 1}$?

[N] Does the system $\dot{x} = \mu x - 2x^2$ have a **subcritical Hopf bifurcation** in the μ, x plane at $\mu = 0$?

[N] **Intermittence** happens during random walk

[N] Strange attractor is the one which repels trajectories from itself **to infinity**

[N] The proof of the Feigenbaum's theorem of ratio universality **is simple**.

[N] The linear and nonlinear stability of fixed points sometimes are two separate things. However, in most mechanical systems, for instance in R3B, linear stabillity **always** implies nonlinear (large amplitude) stability. One example is the motion around the **co-linear** Lagrange points.

[N] Second order integrator algorithm has computational error proportional to the square of the **number of steps** taken to cover a given interval of independent variable.

[Y] In a 2-D linear system, if a fixed point has two imaginary eigenvalues of opposite signs, is the point a center?

[N] Trajectories that start and end at the same nodes are called **separatrices**

[N] In a 2-D linear system, let τ be the trace of the evolution matrix A, and the Δ its determinant. Are the saddle points (unstable fixed points) of a system found with $\tau < 0$ **and** $\Delta < 0$?

[N] Poincare maps are a way to look for chaos in **1, 2**, or multi-dimensional dynamical systems

[Y] Turbulent jets have a universal opening angle. If you double the speed of a water jet injected into a swimming pool, the opening angle will not change. Similarly, if you inject a heavier fluid into lighter, the opening angle remains constant.

[N] Roessler system has fewer non-linear terms, and therefore **lower dimensionality** than Lorenz system

[N] Newton's root-finding method is superstable. By definition, it means that a **constant fraction** of the deviation from the fixed-point disappears in every iteration.

[Y] Chaotic motion (for instance, atmospheric motion) is non-periodic by definition.

[Y] If a dynamical system, for instance a stock market, allows you to use it to gain advantage in a systematic (secular) manner, then that system cannot be perfectly random.

[Y] A tent map is unimodal and exhibits bifurcations

[N] After the Van Der Pol oscillator settles into an asymptotic cycle, it **can be** easily perturbed and destroyed.

[Y] The reason that physical systems often show a close similarity to simple 1-d iterated maps in mathematical dynamical systems (which do not have a continuous variable like time) is mostly based on the fact that the Poincare maps or the Lorenz maps of many physical systems look like unimodal maps, which may exhibit period doublings and chaos.

[N] Cantor flake becomes **longer** by 1/3 in every step of its construction

[N] Koch curve **loses** 1/3 of its points in every step of its construction. In the end it has very few points, in fact a countable number of them.

[Y] Julia set is a map of initial values of a discrete map, on a complex plane.

[N] Many maps undergo bifurcations according to a universal pattern: there is a universal constant $\delta \approx 4.669$, which is a limit as $n \to \infty$ of $(\mathbf{r_n} - \mathbf{r_{n-1}})/(\mathbf{r_{n-1}} - \mathbf{r_{n-2}})$, where r_n is the n-th value at which the bicurcations happen, as the parameter r changes continuously. The condition for this to be true is that the map $x_{n+1} = f(x_n)$ is bimodal.

[N] Fractal is an object that has a **strict** self-similarity: some part of it resembles closely the whole fractal

[N] Feigenbaum discovered that the distance betwen the **fixed** points of a dynamical system decreases in geometrical progression, by a certain universal factor.

[N] Divergence of a vector field representing the phase space flow in a chaotic system **is always** negative.

[Y] Chaos does not happen in the mapping $x_{n+1} = \sin x_n$.

[N] Neural networks can have linear or nonlinear connections. Linear networks learn **faster and better** than nonlinear networks (it is easier to finding the fixed points by local linear analysis than the full non-linear analysis).

[Y] Lorenz map is contracting the phase space volume of a cloud of starting points.

[Y] Tacoma bridge collapsed because vortex shedding frequency was in resonance with the natural flexing mode frequency, and the bridge started swinging violently (non-linearly) in a strong wind.

[Y] Dynamics of airplane's flight led Zhukovsky (Joukowsky) to conclude that gliding flight has naturally decreasing oscillations, later misnamed phugoidal oscillations.

[N] Steady glide angle (with respect to horizon) in radians is **proportional** to the lift-to-drag ratio L/D.

[Y] Phugoidal oscillations in flight of an airplane or glider, represented in the phase plane, are a stable spiral, and are damped significantly after a few periods of oscillation. Feedback mechanisms may shorten this natural damping further.

[Y] In a multidimensional random walk, the path is continuous but the curve lacks unique tangent directions.

[N] If a system is **covered by** a cobweb, then it is **stable**

[N] Feigenbaum has found that the ratio $(\mathbf{b_n} - \mathbf{b_{n-1}})/\mathbf{b_{n-1}}$ where b_n is the n-th bifurcation point parameter, for large n is approaching asymptotically a universal a value of 4.669...

[N] If a **sum** of two eigenvalues of a Jacobi matrix at a fixed point of a 2d system is negative, then the point is a saddle

[N] There are **no** higher n-cycles in **iterated** maps than n=4. Regularity is lost and the system transitions into chaos at $n > 3$.

[N] The double pendulum's largest Lyapunov exponent is **always** positive

[Y] 2-soliton collision leads to phase shift but no change of the final shape of either soliton

[Y] Julia Set is obtained from the quadratic recurrence equation $z_{n+1} = z_n^2 + C$ but unlike the Mandelbrot Set, the Julia set keeps the complex C value fixed while varying the z value.

[N] System $\dot{x} = \mathbf{x^4} + \mathbf{2x^2} - \mathbf{const}$. is called a double-well or bistable system.

[Y] This is the van der Pol equation: $\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$. If the amplitude x^2 is large ($x^2 \gg 1$), then the van der Pol oscillator is damped, for much smaller than 1 values, on the contrary: van der Pol oscillations increase.

[Y] Both types of Hopf bifurcations involve tightly wrapped spirals.

[N] Infinite period bifurcation is the one in which for a certain value of parameter of the system, the amplitude grows to **infinity**, as soon as the motion becomes **periodic**.

[N] Exponential divergence of nearby trajectories is a **suffient** condition for chaos

[Y] Subcritical bifurcation leads to abrupt changes of the behavior

[Y] In order to be called chaotic, the system must still be deterministic (no noisy, or truly randomly varying inputs driving the evolution)

[N] Roundoff error of the processor typically shows the characteristics of a **directed** random walk. Therefore, to destroy, say, 6 accurate digits in a result, a CPU would need to do $\sim \mathbf{10^6}$ arithmetic (floating point) operations.

[N] Classcal physics is **deterministic**, i.e. given the inital conditions, we can always find a unique future trajectory of the dynamical system represented by ODEs.

[Y] A driven physical pendulum can exhibit hysteresis

[Y] The 1-D pendulum equation $m\ddot{x} + \gamma\dot{x} + kx = F\cos(\omega t)$ where $F = const \ll 1$, and $\gamma \ll 1$, is an equation of a weakly damped, weakly driven pendulum described by spring constant k and mass m. Its solutions can be non-oscillatory, for instance can increase without bound for some parameter ω values.

[N] If the tent (of a tent map) is too **flat** then the fixed point is **unstable**

[N] The Lorenz attractor has dimensionality 3, however, there **is** a way to write a similarly behaving, chaotic dynamical system, in dimension **2**.

[N] Aitken process best accelerates the convergence of a series that has an asymptotic exponential-decay approach to a fixed point. Convergence of **all** other series is also accelerated.

[N] Determinant Δ of a 2-D linear system's matrix is a **sum** of the two eigenvalues. Thus, whenever the $\Delta < 0$ at a fixed point, it's a saddle point.

[N] Poincare-Bendixon theorem can prove that a system has **no** periodic orbit.