Final exam PHYD38: Nonlin. Phys and Chaos, 20 Apr 2017

Final is worth 35 percent of course mark, 20 for problems, and 15 for quiz. The weights for which each particular problem are indicated.

1 Written problems.

1.1 [5 pts] Henon map

$$x_{n+1} = y_n + 1 - a\,x_n^2$$

$$y_{n+1} = b\,x_n$$

A. Find all the fixed points of the this discrete map and show that they exist only if $a > a_0$. Find a_0.

B. Calculate Jacobian matrix and its eigenvalues.

C. A fixed point is linearly stable if and only if all eigenvalues of the Jacobian have $|\lambda| < 1$. Determine the stability of fixed points of Henon map, as a function of a and b.

D. Show that one of the points is always unstable

C. Show that the map has a 2-cycle for $a > a_1 = (3/4)(1-b)^2$.

1.2 [4 pts] A nonlinear 2-D system

$$\dot{x} = y - x^2$$

$$\dot{y} = y(a - x - y)$$

where a is a real parameter.

Draw the nullclines, find all the fixed points, classify them and study their nature using eigenvalues and eigendirections. Interpret the values of eigenvalues you find in terms of type of motion (exponential decay, exponential divergence, circulation at a given frequency, etc).

Sketch a plausible phase portrait of the system for the case of $a = 2$.

Are there any bifurcations? If yes, plot a bifurcation diagram.

1.3 [2 pts] Sierpinski sponge

Take a cube (Sierpinski cube) and remove one corner cube that is 1/27th of its volume, ($3^3 = 27$), then continue the proces ad infinitum at smaller scale. Sierpinski sponge is a limit of that procedure, and it is a 3-d version of a Sierpinski carpet in 2-d. Show that it has zero volume, and is a fractal by finding the similarity and box dimensions and showing that they are not whole numbers.

2 Quiz - circle Y (yes) or N (no)

If you circle N, also circle at least one word that does not fit in that sentence. You earn points only if the right reasons for sentence not being correct are given). Ignore simple typos.

[Y...N] System $\dot{x} = x - x^3$ has a potential $V = -(1/2)x^2 + (1/3)x^3 + const.$

[Y...N] There is a theorem for nonlinear centers of conservative 2-D systems: If a conserved quantity E($\mathbf{x}$) exists and has a minimum at an isolated fixed point on a plane $\mathbf{x}$, then all trajectories close to this fixed point are closed.

[Y...N] Can a 4-dimensional system not have oscillations?

[Y...N] Chaotic oscillations of forced damped double pendulum are sometimes periodic

[Y...N] Secular term is a periodic part of solution (e.g. one of resonantly forced oscillator system that grows a perturbation time).

[Y...N] Does the system $\dot{x} = 4r - x^2$ have a saddle-node (turning-point) bifurcation at $r = 1$?

[Y...N] Is the critical slowing down occurring in the system $\dot{x} = 2rx - x^3$ for $r = 1$?

[Y...N] Does the system $\dot{x} = \mu x - 2x^2$ have a subcritical Hopf bifurcation in the μ, x plane at $\mu = 0$?

[Y...N] Both types of Hopf bifurcations involve tightly wrapped spirals.

[Y...N] Infinite period bifurcation is the one in which for a certain value of parameter of the system, the amplitude grows to infinity, as soon as the motion becomes periodic.

[Y...N] Exponential divergence of nearby trajectories is a required feature of what we call chaos

[Y...N] Intermittence of motion is required feature of what we call random walk

[Y...N] Strange attractor is the one which repels trajectories from itself to infinity

[Y...N] The proof of the Feigenbaum's theorem of ratio universality is surprisingly simple.

[Y...N] The linear and nonlinear stability of fixed points sometimes are two separate things. However, in most mechanical systems, for instance in R3B, linear stabillity always implies nonlinear (large amplitude) stability. An example os the motion around the triangular Lagrange points.

[Y...N] The reason to strive for optimum timestep control is not only speed of the computation but mainly its numerical accuracy.

[Y...N] In a 2-D linear system, if a fixed point has two imaginary eigenvalues of opposite signs, is the point a center?

[Y...N] Trajectories that start and end at the same nodes are called separatrices

[Y...N] In a 2-D linear system, let τ be the trace of the evolution matrix A, and the Δ its determinant. Are the saddle points (unstable fixed points) of a system found with $\tau < 0$ and $\Delta < 0$?

[Y...N] Poincare maps are a way to look for chaos in 1, 2, or multi-dimensional dynamical systems

[Y...N] Box dimension is number q in the relation $m = r^q$, connecting the number of copies of an object fitting within area of r-times larger radius or linear size (zoomed r times).

[Y...N] Roessler system has less non-linear terms and therefore lower dimensionality than Lorenz system

[Y...N] Chaotic motion (for instance, atmospheric motion) is never periodic.

[Y...N] If a dynamical system, for instance a stock market, allows you to use it to gain advantage in a systematic (secular) manner, then that system can be proven to be non-random.

[Y...N] Is an iterated map $f_{n+1} = \exp(-f_n) \ln f_n - const$ unimodal and does it have bifurcations?

[Y...N] After the Van Der Pol oscillator settles into its asymptotic form of oscillations, its time period depends sensitively on the amplitude

[Y...N] In financial markets, is is advantageous to analyze log(price ratio) rather than juzt day-to-day price ratio or price increment, since the particular levels of indices are not important, prices depend on the currency and so on.

[Y...N] The reason that physical systems show often close similarity to simple 1-d maps in mathematical dynamical systems (which do not even contain a continuous variable like time) is mostly based on the fact that the Poincare maps or the Lorentz maps of many the physical systems are looking like tent maps, ready to iterate chaotically or show period doublings before that.

[Y...N] Cantor set lengthens by 1/3 in every step of its construction

[Y...N] Koch curve loses 1/3 of its points in every step of its construction. In the end it has very few points - it is countable.

[Y...N] Sierpinski carpet has infinite area

[Y...N] Many maps undergo bifurcations according to a universal pattern: there is a universal constant $\delta \approx 4.669$, which is a limit as $n \to \infty$ of $(r_n - r_{n-1})/(r_{n-1} - r_{n-2})$, where r_n is the n-th value at which the bicurcations happen, as the parameter r changes continuously. The condition for this to be true is that the map $x_{n+1} = f(x_n)$ is bimodal.

[Y...N] Fractal is an object that has a strict self-similarity: some part of it (e.g. every quarter of the length of Koch curve) resembles closely the whole fractal

[Y...N] Chaotic system must be nonlinear.

[Y...N] Feigenbaum discovered that the distance betwen the fixed points of a dynamical system decreases by a certain universal factor.

[Y...N] Chaotic system must be intermittent.

[Y...N] Intermittency does not happen in sine map $x_{n+1} = sin(x_n)$

[Y...N] Neural networks divide into linear and nonlinear networks. Linear networks learn faster than nonlinear networks, in analogy to a smaller complexity of finding the stability of dynamical systems by local linear analysis around fixed points, as opposed to full non-linear one.

[Y...N] Lorenz map is contracting the phase space, i.e. phase-space volume of the cloud of starting points (representing individual solutions)

[Y..N] Markov Chain is a numerical approach to solving variational problems such as chain curve - the shape of a freely hanging chain made of many links