Midterm exam PHYD38 2024 Preparation

1 Written problem: 2-D nonlinear system with bifurcation

Using approaches discussed in the lectures and assignments, study as thoroughly as you can the system

$$\dot{x} = -y/2 - x^2$$
$$\dot{y} = cx - x^3$$

Here, x and y are variables and c parameter, all of them real-valued.

A. In this point, assume $c > 0$. Sketch the nullclines.

ANS: x-nullcline is $y = -2x^2$, y-nullcline is the sum of three vertical lines: $x = 0$ (y-axis) and two vertical lines at $x = \pm\sqrt{c}$.

Find the fixed points and their stability. Find eigenvectors. From eigenvalues, find the timescales of convergence/divergence to/from the saddles, if any. for centers, find the frequencies of rotation around them and the corresponding periods.

SOLUTION

Three fixed points are (0,0),$(\pm\sqrt{c}, -2c)$. Jacobi matrix's upper row is: $-2x$, $-1/2$. Lower row is: $c - 3x^2$, 0.

At the first point, (0,0), the matrix has $\tau = 0$, $\Delta = c/2 > 0$, hence it's a linear center. Rotation is counter-clockwise. Of course, circles are closed only in a small vicinity of (0,0).

At the second and third points, $x = \pm\sqrt{c}$. Substituting into the Jacobi matrix, we derive its $\tau = \mp 2\sqrt{c}$, and $\Delta = -c < 0$. Those two fixed points are saddles with the following λ's: $\lambda_1 + \lambda_2 = \tau = \mp 2\sqrt{c}$, and $\lambda_1\lambda_2 = \Delta = -c < 0$, which we work out below.

Let's take the point $(\sqrt{c}, -2c)$. Solving for $\lambda_{1,2}$ and eigenvectors, we obtain:

$$\lambda_{1,2} = (-1 \pm \sqrt{2})\sqrt{c} = (+0.41, -2.41)\sqrt{c}$$

(one positive, one negative of bigger absolute value). Timescale of exponential divergence/convergence is $1/|\lambda| = c^{-1/2}/|-1 \pm \sqrt{2}| = (2.41, 0.41)c^{-1/2}$.

The corresponding eigendirections are obtained from the equation $(A - \lambda I)w = 0$, where $w = (w_x, w_y)$ is an eigenvector, in which we can assume $w_x = 1$ to calculate w_y, since we only care about the correct eigendirection of divergence/convergence):

$$y = 2\sqrt{c}(-1 \mp \sqrt{2})x = (-2.41, +0.41)\sqrt{c}\,x.$$

The slope is positive when the eigenvalue is large and negative (eigendirection of fast convergence to saddle point). Slope is negative when the eigenvalue is positive (eigendirection of slower divergence from the saddle point). The slope depends on $c > 0$.

At the point $(-\sqrt{c}, -2c)$, we have:

$$\lambda_{1,2} = (1 \pm \sqrt{2})\sqrt{c} = (+2.41, -0.41)\sqrt{c}$$

(one positive, and one smaller negative). Timescale of divergence/convergence are again the inverted lambda's, equal (0.41, 2.41). The corresponding eigendirections are:

$$y = 2\sqrt{c}(1 \mp \sqrt{2})x = (-0.41, +2.41)\sqrt{c}\,x.$$

The slope is negative when the eigenvalue is positive (divergence), and positive (of smaller absolute value) when the eigenvalue is negative (convergence).

B. Sketch the phase portrait of the system for $c = 1$.

The picture of the whole phase space is somewhat similar to the one which appears on p. 166 of the Strogatz book, only turned around (x swapped with y).

From the preceeding description of directions and slopes, the direction of the heteroclinic trajectory on the $y > 0$ side of the figure is from he right saddle to the left saddle, also the circulation around the center at the origin is counterclockwise. There is also a heteroclinic trajectory (saddle connection) below $y = -2c$.

C. In this part, consider arbitrary real c as a bifurcation parameter. Find the critical value of c. Sketch the bifurcation diagram: x^* as a function of c, and name the bifurcation. Use dashed line for branches consisting of unstable fixed points and solid line for sinks and centers. Name the bifurcation.

ANS: the bifurcation lines are a 3-pronged fork in (c, x^*) plane.

For $c < 0$ there are no real solutions to the $c - x^2 = 0$ equation, so the x-nullcline is one line, a vertical axis $x = 0$. The two saddles appear/disappear (bifurcate) at the critical parameter value $c = 0$. The center $(x^*, y^*) = (0, 0)$ is still a fixed point, but since $\Delta = +c/2$ there, for negative c the origin becomes a saddle point (unstable).

For positive c, only the fork's handle ($x^* = 0$) is stable, while the $x^* = \pm\sqrt{c}$ prongs are unstable (saddles).

Q: Is linearization method (such as the Jacobi matrix method) enough to make sure that center (i.e., linear center) are stable in Lyapunov sense?

ANS: No, linear analysis may be insufficient in case of cycles, but in our case there are bounding heteroclinic trajectories (heteroclinic lines) that cannot be crossed, so the center and its immediate surroundings is Lyapunov stable.

2 Quiz

Circle Y or N = yes or no. If you answer N, also circle at least one word of the statement that is wrong (otherwise you'll get no credit even if your answer happens to be correct). Please ignore typos and/or language errors.

SOLUTIONS: boldfaced fragments are wrong, but circling other words may be OK too.

[Y/N] Application of Poincare-Bendixon theorem to the van der Pol oscillator shows that it converges to a stable orbit. [We've shown in the tutorial that the anwer is Y. However, it was too suggestive of the answer "N, it's not P-B theorem, it's Dulac's criterion – so ok, this will be one of the "free" points.]

[N] Linear dynamical systems have eigenvectors **always aligned** with nullclines at fixed points

[N] For $a < 0$, system $dx/dt = a - x^3$ has one fixed point, which is stable, and for $a > 0$ one **unstable** point

[Y] The system $dx/dt = x^2$ has one fixed point, which is half-stable

[Y] In a Runge-Kutta method of solving the first-order ordinary differential equation(s) of a dynamical system numerically; each time step is divided into many substeps or iterations, and one evaluation of the right-hands sides of the evolution equations is done per substep.

[N] The logistic equation reads: $\dot{N} = KN - \gamma N^2$. Here, K is a called **carrying capacity** (limit of growth of N), and γ is a constant specifying the adverse effect of crowdiness on population.

[N] Critical slow-down happens when a solution of a dynamical system **becomes constant** (stops evolving) at a finite time t.

[N] Does the system $\dot{x} = r - x^2$ have a **transcritical** bifurcation at $r = 0$?

[Y] Supercritical Hopf bifurcation happens when a damped oscillation (inward spiral) turns into a growing oscillation (outgoing spiral in phase space)

[Y] Does the system $\dot{\omega} = -1 - b \sin \omega$ have a saddle-node bifurcation at $b = 1$?

[Y] Systems that exhibit subcritical pitchfork bifurcation, can exhibit hysteresis, but they must contain more nonlinearity (higher powers of x) than this pitchfork system: $\dot{x} = rx + x^3$

[N] In a 2-D linear system, if a fixed point has two **real** eigenvalues of opposite signs, is the point a center?

[N] Trajectories that start and end at the same node are called **nullclines**

[N] Index theory says that index $I = +1$ of a fixed point proves that it is stable.

[Y] Index theory says that index $I = -1$ of a fixed point proves that it is unstable.

[N] Do we say that a fixed point is Liapunov stable if and only if all trajectories that start sufficiently close to it, do not depart from it **to infinity**?

[Y] Lorenz attractor contracts the phase space (divergence of the flow is negative), therefore is sometimes called a dissipative system, in analogy to physical systems with damping/friction, which do the same.

[N] Lorenz system is the simplest system of **linear** equations exhibiting chaos: is 3-dimensional (has 3 independent variables, for whose time derivatives Lorenz gave a set of linear equations)

[Y] First order perturbation theory using series expansion in a small parameter ϵ is prone to show spurious problems at large time, since the driving (r.h.s.) terms may be in resonance with the natural oscillation frequency of the system.

[Y] One of the examples of successful two-timing perturbation theory is its application to van der Pol equation