MIDTERM EXAM PHYD38: NONLIN PHYS & CHAOS, 2024

1 Written problems with solutions

1.1 [20 p.] Problem 1

Analyze graphically, sketch and name the bifurcations in the bifurcation diagram of a 1-D dynamical system with real parameter r:

$$\frac{dx}{dt} = rx - x(1-x)^2$$

Without solving the differential equation, state the asymptotic value of variable x at $t \to \infty$ for the following 3 different starting values at $t = 0$:
(i) $r = 0 = const.$, $x = 4$
(ii) $r = 1/2$, $x = -4$
(iii) $r = 2$, $x = 1$.
Hint: If it's easier, you can draw $r(x^*)$ first, then re-draw it as $x^*(r)$.

Solution

We find all the fixed point positions in (x, r) plane [we skip the asterisk in x^*. The curves we will draw are the $r(x^*)$ curves.]

From $dx/dt = 0$ we get either $x = 0$ or $r = (1-x)^2$, i.e. the sum of two curves: a vertical axis in (x, r) plane and a parabola with the minimum at $x = 1, r = 0$, which is crossing the vertical axis at $r = 1$ point.

Graphical analysis of the flow (which always happens along the x-direction for various $r = const$ values, will show which branches of the curves are stable and which unstable. The direction of motion between the lines of fixed points is easy to figure out from the r.h.s. of the original ODE.

The first point of intersection of $x = 0$ and $r = (1-x)^2$ curves in (x, r) plane is (0,1). As we move up along the r-axis from negative values to $r = 1$, the fixed points on it are stable (because $\dot{x}$ for small $x > 0$ is negative and vice versa. The $x = 0$ solution at $r = 1$ loses stability to the left parabolic branch, extending to the $x < 0$ semi-plane, and itself remains an equilibrium (unstable for $r > 1$). The point of intersection is subcritical bifurcation.

The second interesting point is $(0, 1)$, where moving from left to right (toward larger x^*, along the parabola) unstable branch changes to stable. It is a saddle-node (supercritical or blue-sky) bifurcation at $r = 1$.

Of course you want to draw the rotated bifurcation diagram in swapped variables (r, x^*).
`https://planets.utsc.utoronto.ca/~pawel/PHYD38/IMG_8841.JPG`

1.2 [45 p.] Problem 2

A. [35p.] Characterize the fixed points of the system

$$\dot{x} = y - \frac{x^3}{2}$$

$$\dot{y} = x - 2x^2 y$$

where x and y are real-valued variables.

Sketch nullclines, find all fixed points, study their type, eigenvalues and eigenvectors. To get a more truthful flow field and shapes of trajectries, you may want to choose some straight lines (e.g. eigendirections of a saddle point, or some vertical or horiozontal lines, and sketch at what angle the $(\dot{x}, \dot{y})$ vectors intersect such a line, as x or y changes. Your final goal is to do a good sketch of the flow in the whole (x, y) plane.

B. [10p.] One of the fixed points is a saddle. Using index theory in 2-D systems, judge whether it is possible that at a very large distance from the origin and the 3 fixed points, the flow generally looks like a flow around a saddle point. (What is the index of a curve enclosing all the fixed points of this system?)

Solution

A. We draw curves in the phase plane (x, y). The x-nullcline is

$$y = x^3/2$$

. Vectors of the flow cross it vertically.

The $\dot{y} = 0$ nullcline condition produces a sum of two curves: vertical axis $x = 0$ and hyperbola $y = \frac{1}{2x}$. The flow crosses them horizontally.

x- and y-nullclines cross at three fixed points: (0,0), (0,1/2) and (-1,-1/2).

Standard linear (matrix) analysis of point (0,0) shows a saddle point with one eigenvalue $\lambda = +1$ leading to exponential outflow along $y = x$ line and the other eigenvalue $\lambda = -1$, describing inflow along the $y = -x$ direction.

Around the $(1, 1/2)$ and $(-1, -1/2)$ points, the the linear analysis returns ingoing spirals with the same, complex eigenvalues, and same eigenfunctions; therefore also the same clockwise sense of rotation of flow around the stable sinks. The real part deciding about the rate of exponential approach is $Re(\lambda) = -7/4$, while the imaginary parts are close to ± 1. (More precisely, $\sqrt{15}/4 < 1$.) This means that the spiral has very open arms, namely when time advances by about $\Delta t \approx 2\pi$, the radial distance to the fixed point decreases by as much as a factor $\exp(-7/4\Delta t) \approx \exp(-14 * 3/4) \approx 3 \cdot 10^{-6}$. The spiral does not make even one full term before it approaches extremely closely the fixed point.

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B.The index of any closed curve enclosing 3 fixed points is the sum of indices of individual points: +1+1-1 = +1. Therefore, an arbitrarily large circle surrounding the fixed points has index +1 and cannot be a saddle.

2 Quiz with solutions

Boldfaced words are wrong, but they may not be unique: sometimes other wrong words could have been circled.

[Y] System which has a saddle-node bifurcation can have either: no fixed point, one half-stable point, or one stable plus one unstable point.

[Y] Stable circles in 2-D systems have index +1 or -1 , for both anti-clockwise and clockwise circulation of flow around the fixed point.

[N] One-dimensional systems **cannot** have periodic solutions

[N] Euler method is a method of solving the first-order ordinary differential equation(s) of a dynamical system numerically. In every time step it incurs an error of order Δt, which is why we call it first order method.

[N] The logistic equation reads: $\dot{N} = \kappa N - \gamma N^2$. Here, κ is a called the **carrying capacity** or limit of growth of N, an γ is a constant specifying the adverse effect of overpopulation.

[Y] Critical slow-down happens when a solution of a dynamical system slows down a lot but never really stops evolving in some finite time interval.

[N] System $\dot{x} = rx - x^2$ has a transcritical bifurcation bf at $r = 1$

[Y] Supercritical Hopf bifurcation happens when a damped oscillation (inward spiral) turns into a growing oscillation (outgoing spiral in phase space)

[Y] System $\dot{\omega} = -1 + b \sin \omega$ has a saddle-node bifurcation at $b = -1$

[N] System $\dot{x} = rx + x^3$ does **allows hysteresis**

[N] In a 2-D linear system, if a fixed point has two **real** eigenvalues of opposite signs, that point is a center

[N] Trajectories that start and end at the same node are called **nullclines**

[Y] According to index theory, index $I = -1$ of a fixed point proves that it is unstable

[N] Index theory says that index $I = +1$ of a fixed point **proves that it is stable.**

[N] When there is only one eigendirection at a fixed point of a 2-D system, then the fiex point is **unstable**?

[N] Centers are delicate because the trajectory has to close ideally after one circle (be periodic). A very small nonlinearity of equations destroys that property, and **means that the circle becomes unstable**.

[Y] A system $\ddot{x} = f(x)$ where $f(x)$ is arbitrary function of x but not $\dot{x}$ is both conservative and time-reversible.

[N] Trajectories in the immediate vicinity of a closed cycle are either departing from it, converging to it, or **closed trajectories**.

[Y] Close orbits are impossible in gradient systems (where the right-hand side has potential function), and in systems with positive-definite Liapunov function V that obeys $dV/dt < 0$.

[Y] Poincare-Bendixon theorem guarantees existence of a stable periodic solution in a region of a 2-D phase space that is confining trajectories (does not have any trajectory crossing a bounding closed curve R to toward the outside).

[N] Van der Pol system originated in early experiments with nonlinear electronic components. It leads to **homoclinic oscillations**.

[Y] A bead moves with a lot of friction on a circular hoop that rotates at a constant angular speed ω. The dynamical system has a fork bifurcation at a critical rotation speed.

[N] If there was no air drag force, a glider would move either with or without oscillations of glide angle and velocity. But a glider subject to air drag **cannot** move without such phugoidal oscillations.

[N] A system $d^2x/dt^2 = -x + x^3 - |dx/dt|^2$ is time-reversible (i.e., **conservative**).

[N] Heteroclinic trajectory from A to B **can cross** a homoclinic trajectory from some other point C back to C.