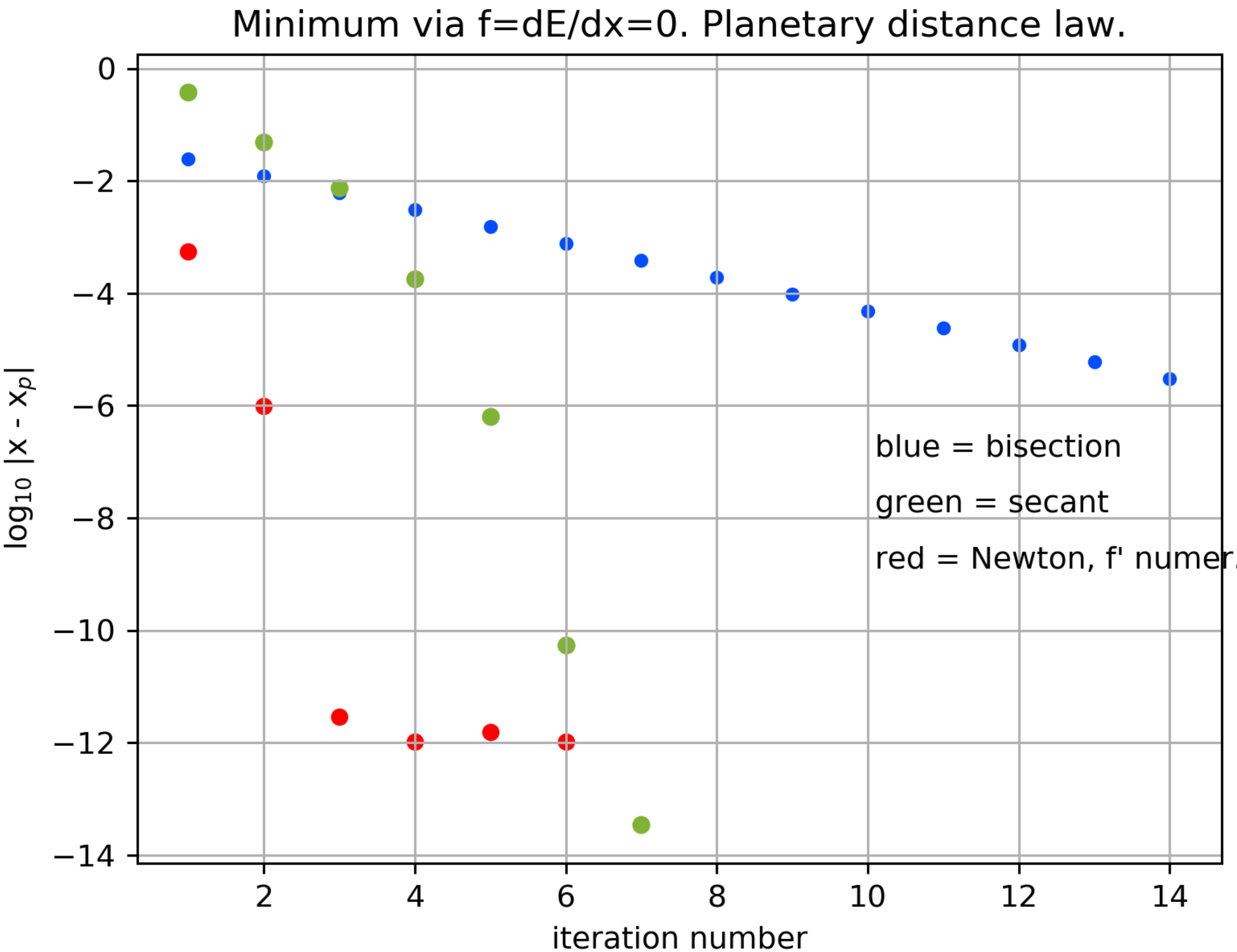
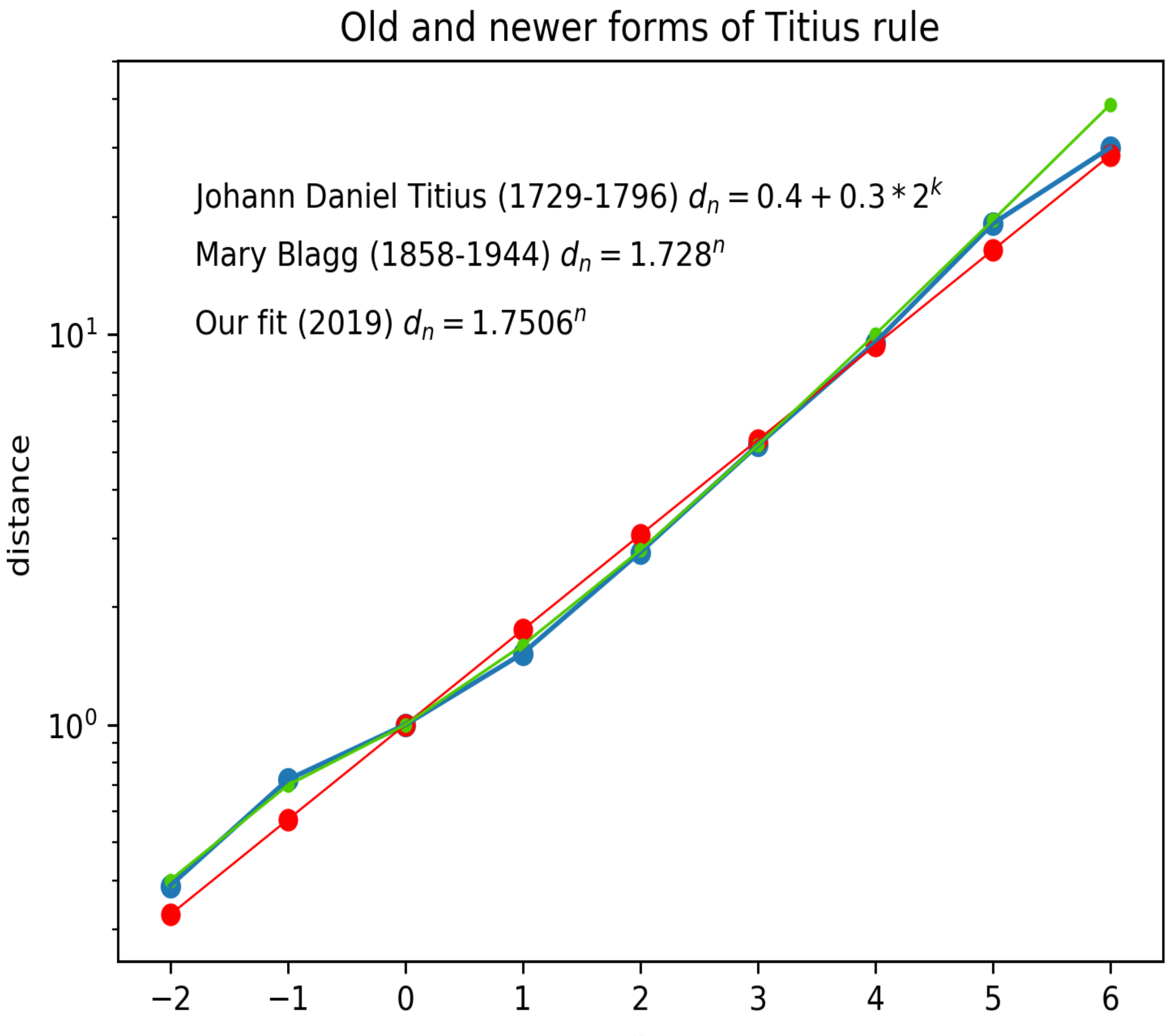
Lecture 10

- Some assignment 3 problems – Titius law
- **Interpolation**
  - Lagrange polynomials and their limitations
  - Splines
- **Ordinary Differential equations**
- Single, 1st order ODE. Sets of 1st order diff.
- ODEs are equivalent to higher order eqs.
- Taylor expansions and integration schemes:
  - derivation of trapezoid and midpoint methods
  - RK4 scheme; comments on multistep methods
  - shooting method in split-point boundary conditions
- Examples of 1st order nonlinear equations:
  - y' = -y cos(x), y' = -y $\sin^3(x)$; accuracy, error drift, convergence
- y'' = -g - unusually simple and accurate 2nd ord. solutions
- chapter 7 in Turner et al (2018)

Titius-Bode rule problem




Old and newer forms of Titius rule
Johann Daniel Titius (1729-1796) $d_n = 0.4 + 0.3 * 2^k$
Mary Blagg (1858-1944) $d_n = 1.728^n$
Our fit (2019) $d_n = 1.7506^n$
distance
10^1
10^0
n
−2
−1
0
1
2
3
4
5
6

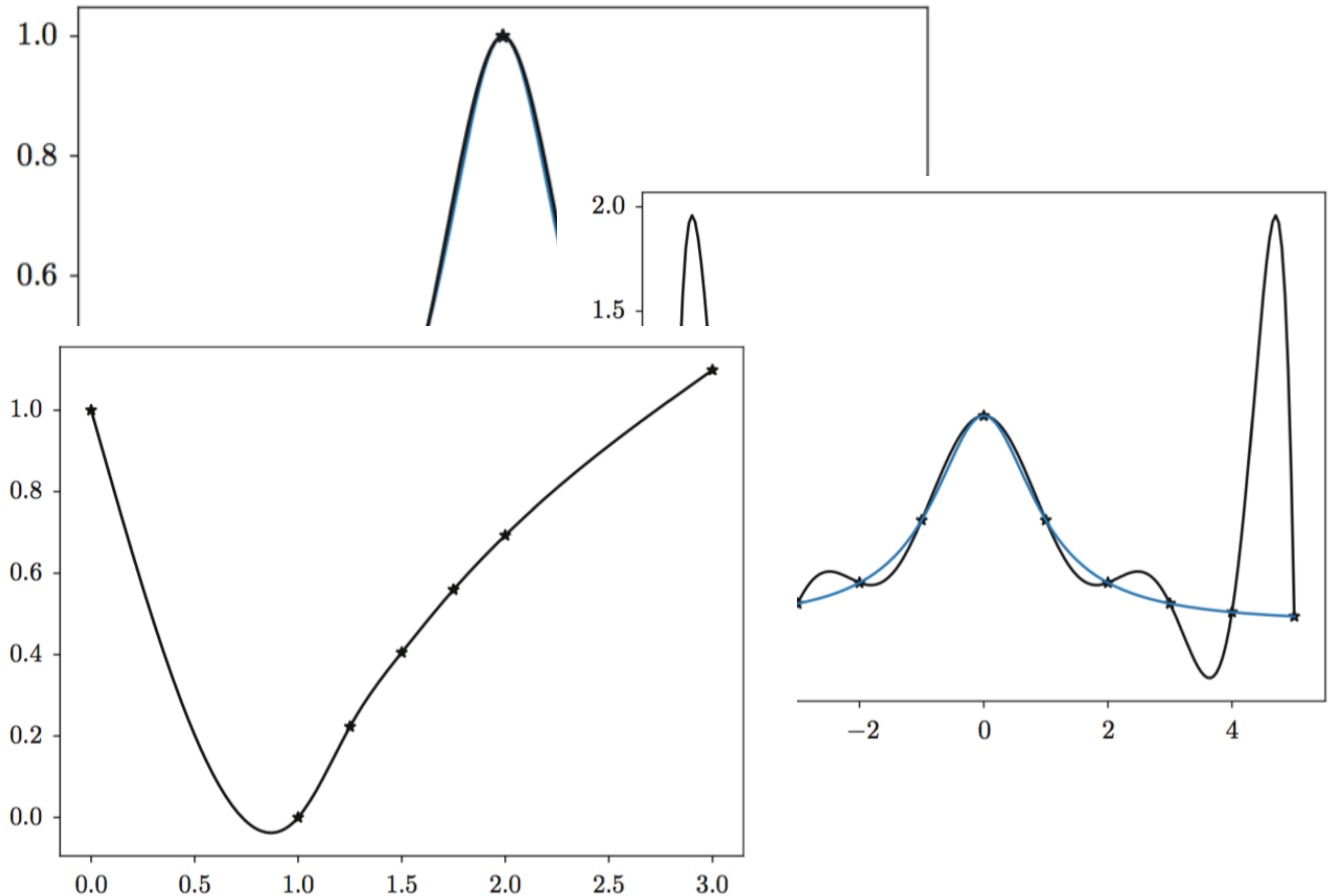
Interpolation

- Turner et al (2018) – read chapter 6, starting with p. 189

- Interpolation by polynomials: **Lagrange polynomials**
  - problem: wiggly polynomials
  - the trouble also explains why extrapolation is difficult
- Interpolation by piecewise polynomials:
  - **Splines**; much nicer than polynomials in general
  - cubic splines have continuous 0^{th}, 1^{st} and 2^{nd} deriv.
  - they require solving a tri-diagonal linear system for constants of piecewise cubic functions.
  - this is a cheap calculation, only O(N) arithm. operations
  - `f = scipy.interpolate.interp1d(X,Y,'cubic')`
  - offers several different interpolation schemes depending on string argument, input: X and Y (numpy arrays of length N). Result is a *function* that you can call with some other array of x's to interpolate & plot:
  - `plt.plot(x,f(x));     plt.show()`

Interpolation, Extrapolation, Splines

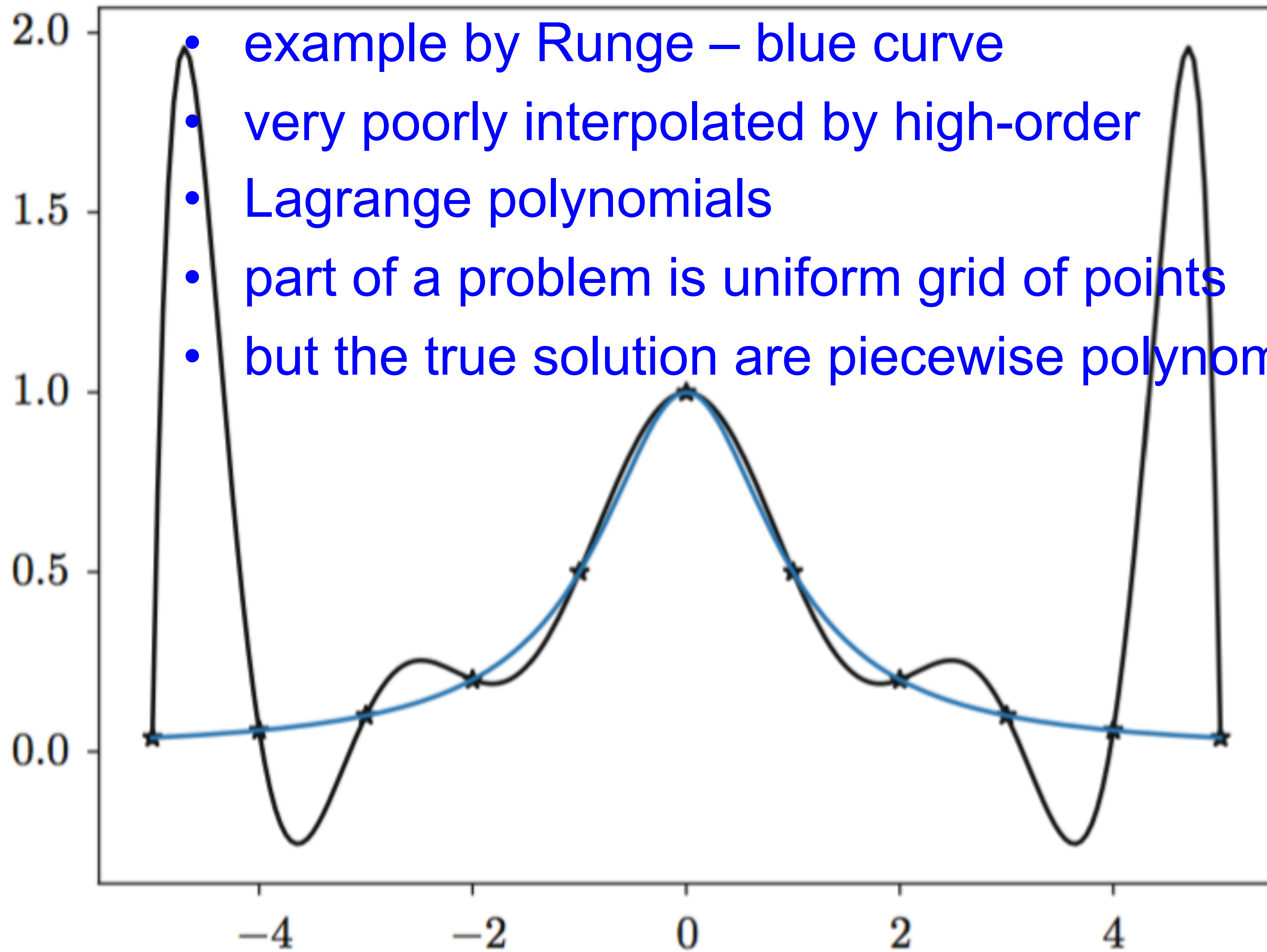
Turner et al. (2018) - textbook

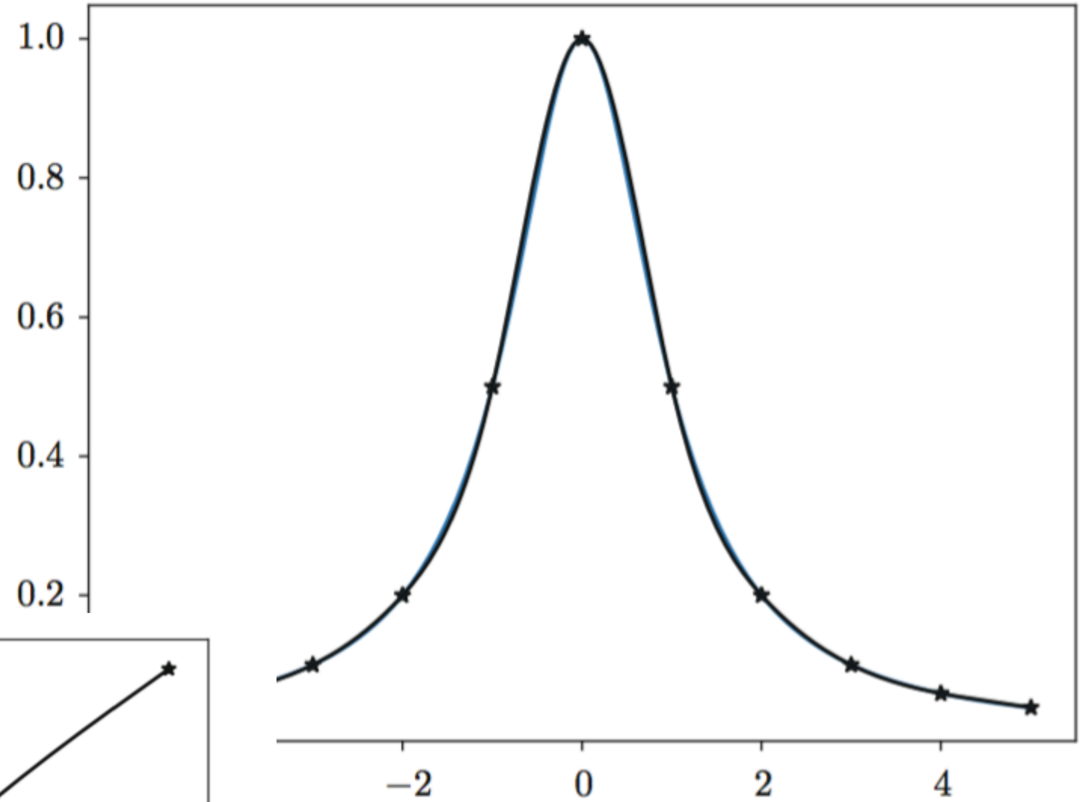


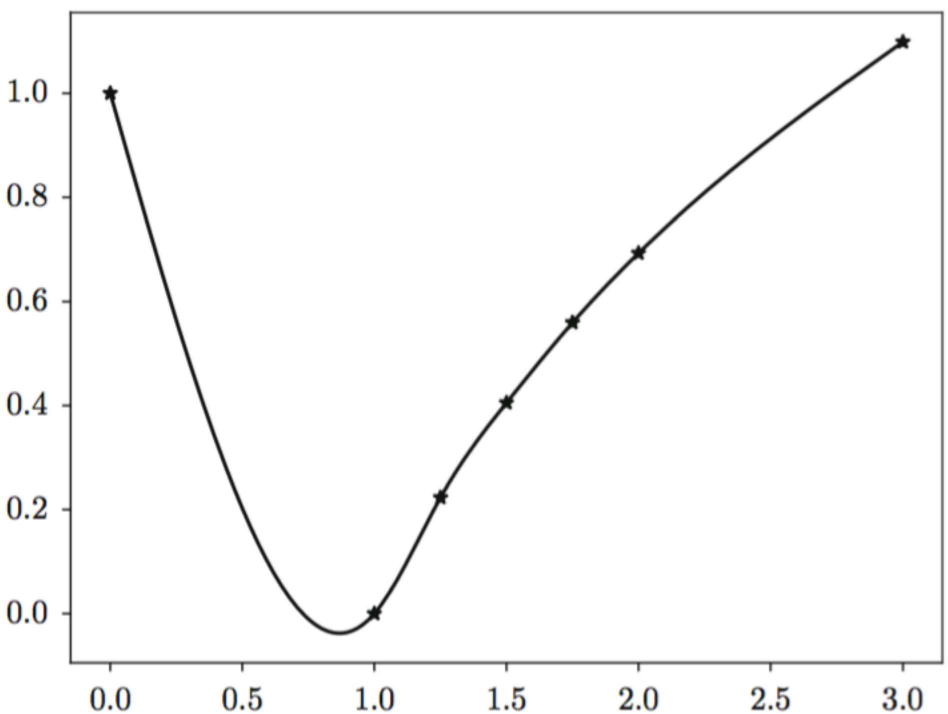
Interpolation, Extrapolation, Splines

Turner et al. (2018) - textbook

- there is a problem with polynomials
- example by Runge – blue curve
- very poorly interpolated by high-order
- Lagrange polynomials
- part of a problem is uniform grid of points
- but the true solution are piecewise polynomials







splines have continuous 1st and 2nd order derivatives across intervals

To get their coefficients, one solves a linear algebraic set of eqs., which has a band structure -> fast solution

Fig. 6.9 Interpolation of the Runge function with a cubic spline

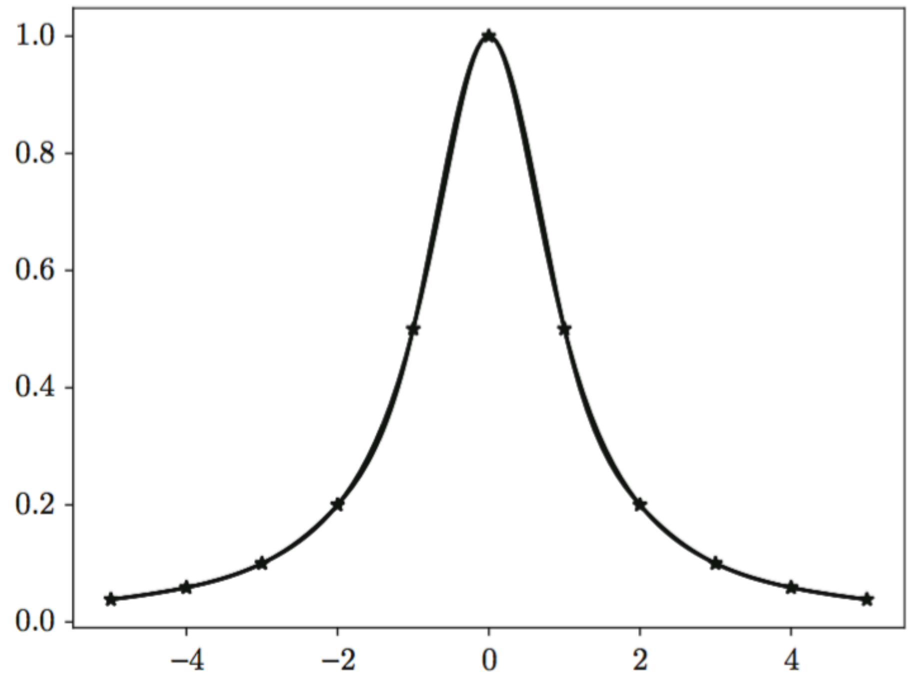
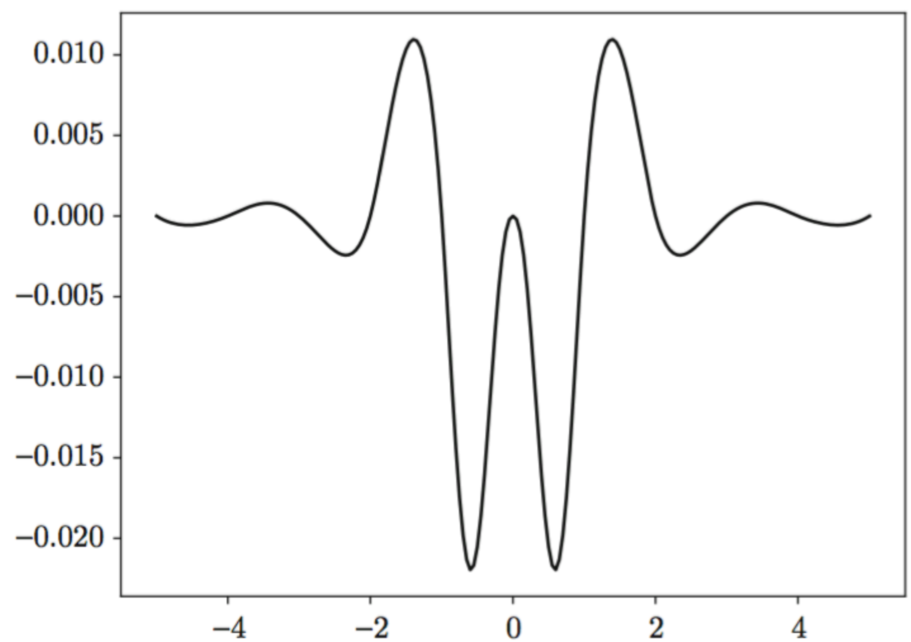


Fig. 6.10 Error of the interpolation from Fig. 6.9



ODEs

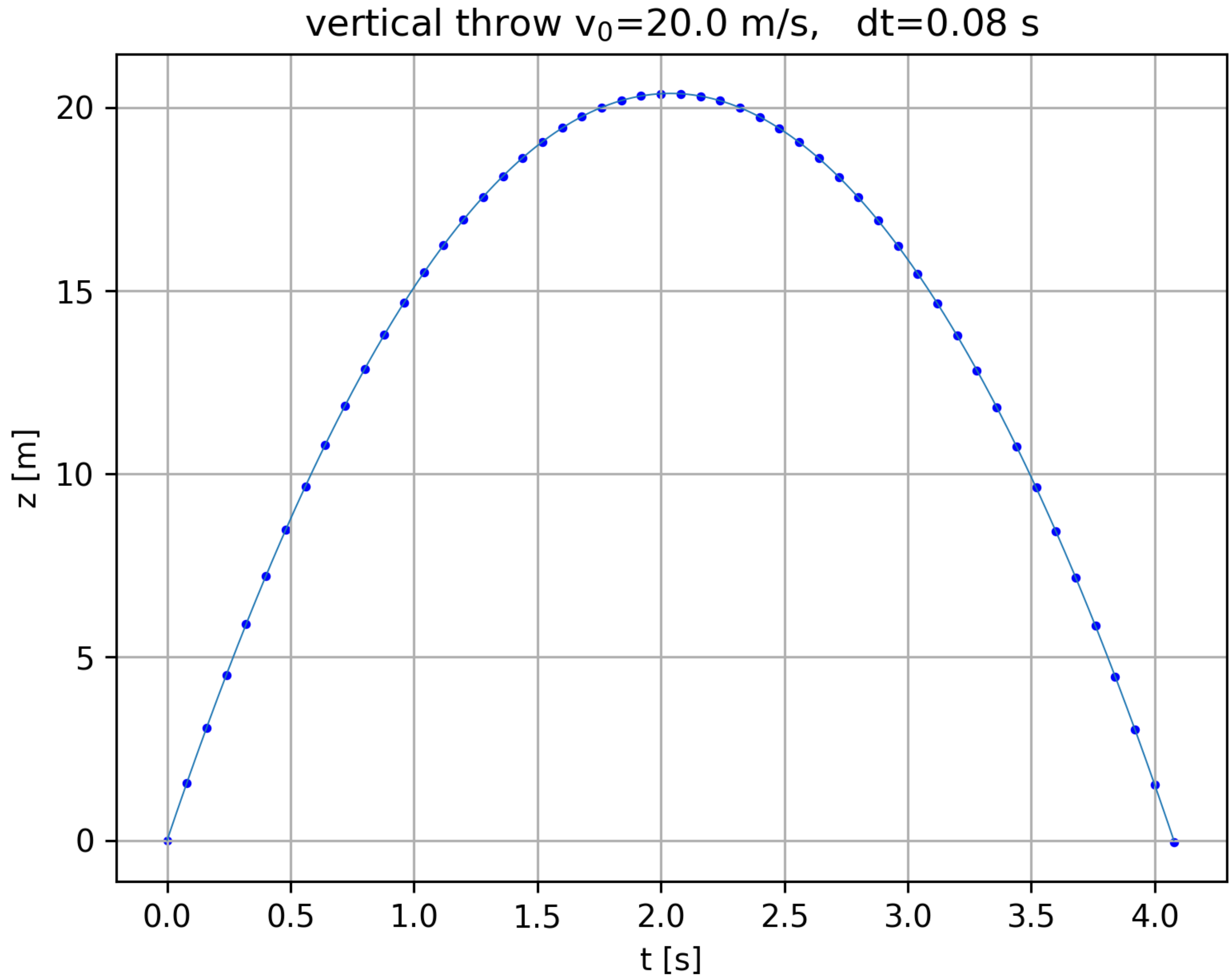
Turner et al. pp. 229+

- $dy/dt = y' = f(x,y)$ - simple first order ODE
- simple 1D 1st order differential equations:
- $y' = -y \sin(x)$
- $y' = -y \sin^3(x)$ initial value problems, B.C.: $y(0)=1$

- $d^2y/dt^2 = f(x,y)$ - simple second order ODE,
- Newtonian, Hamiltonian, and Langrangian dynamics is full of such ODEs, often $f(x,y) = f(y)$ t==x not explicit
- Taylor expansion useful to create integration methods
  - example: trapezoid rule is 2nd order.
- Similarities and differences with definite integrals of functions
- Basics of drag forces and their implementation
  - example: throwing a ball in vacuum vs. in air

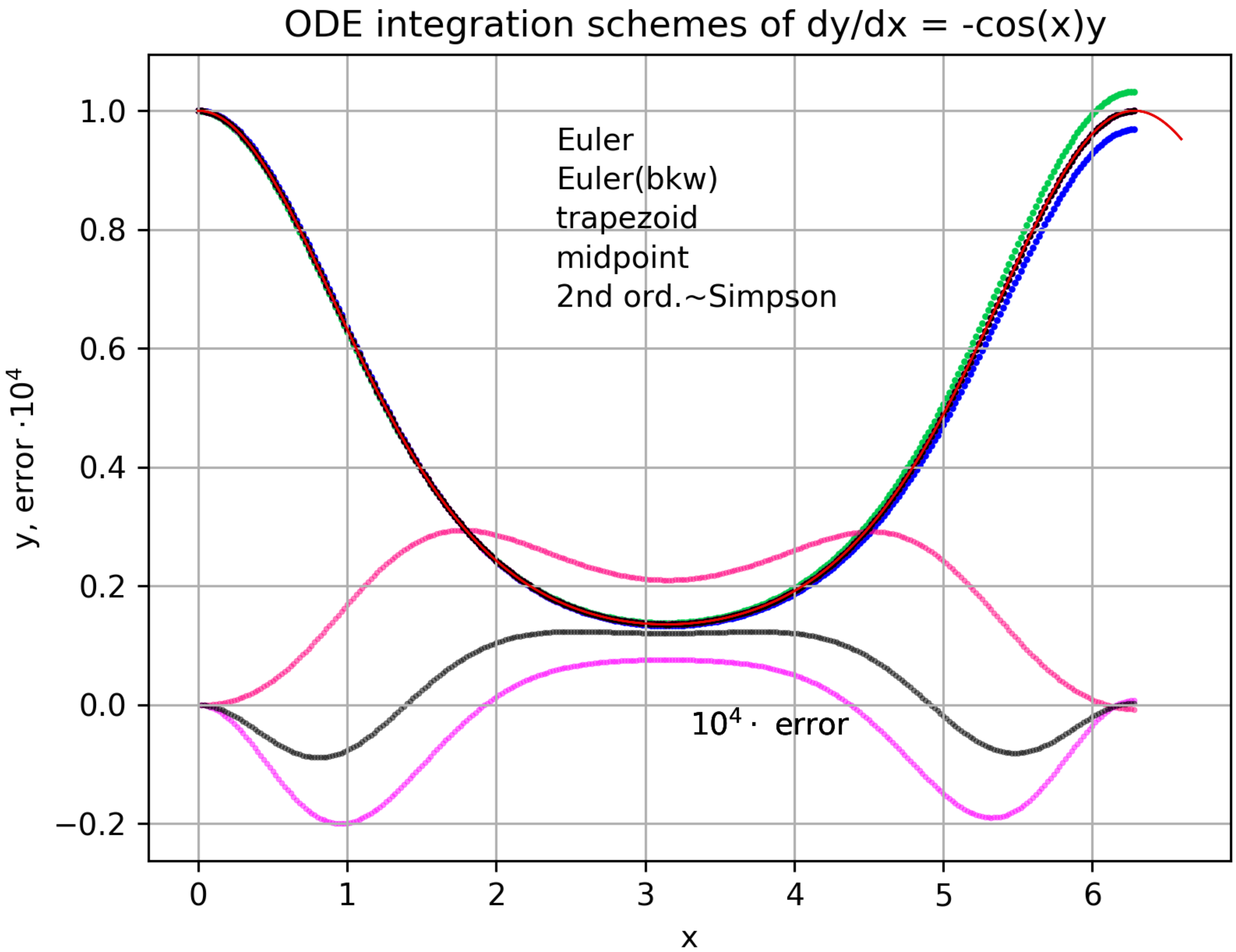
vertical throw in vacuum – 2^{nd} order 1D ODE

Trapezoidal rule. Equations so simple (no higher order derivatives
that this scheme returns zero errors with sizeable step dt.

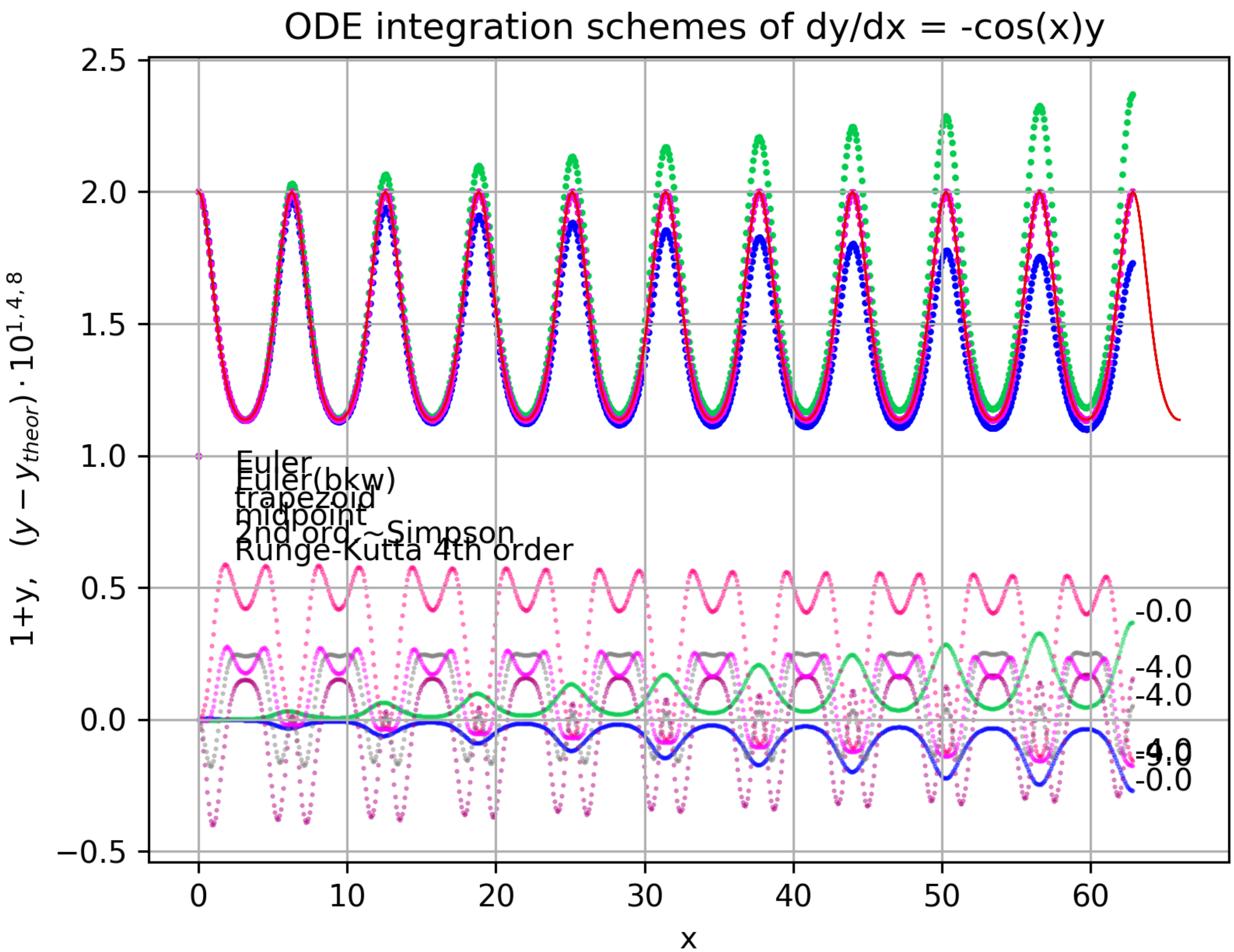


diff1-throw-6.py

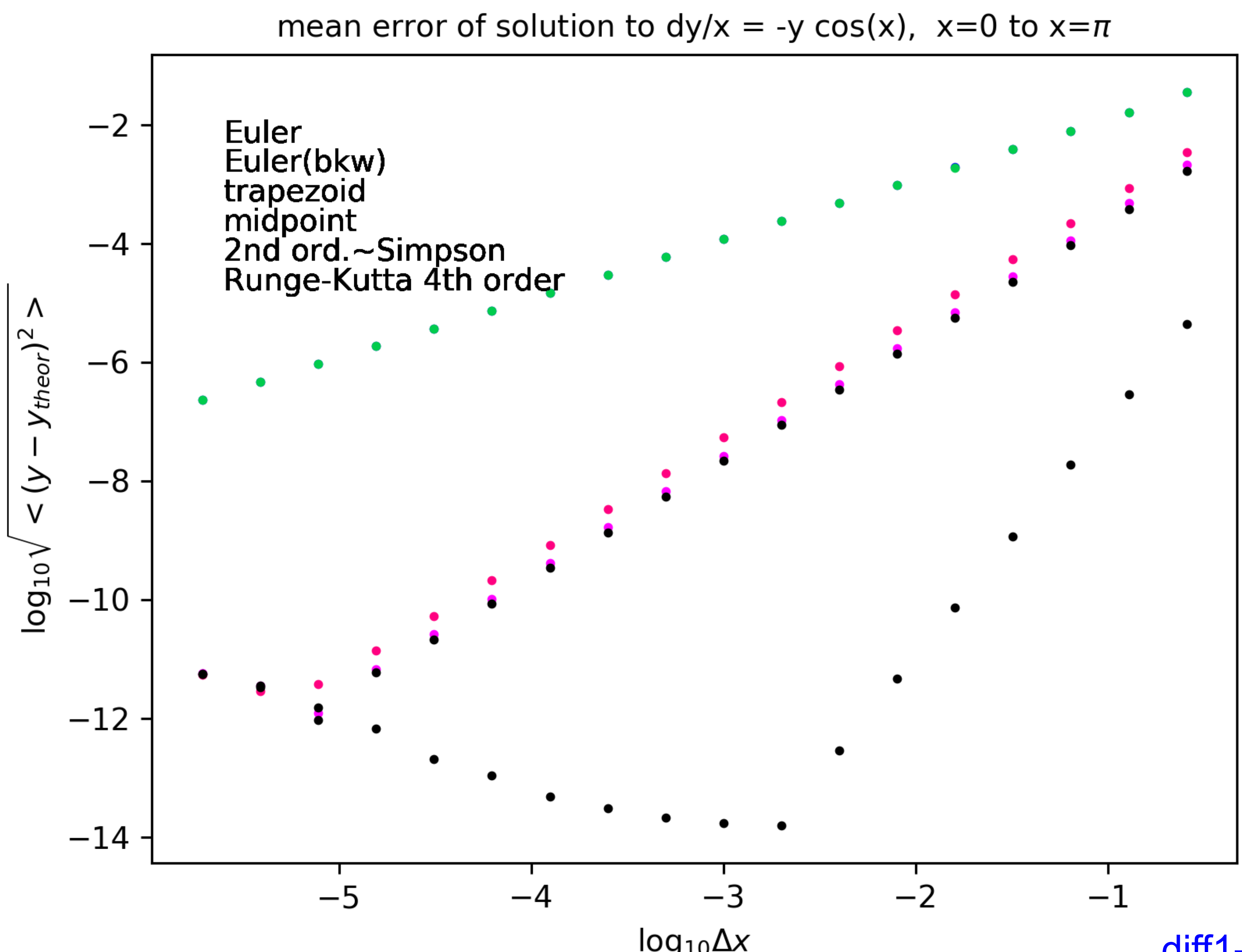
diff1-ex3.py


ODE integration schemes of dy/dx = -cos(x)y
Euler
Euler(bkw)
trapezoid
midpoint
2nd ord.~Simpson
10^4 · error
y, error ·10^4
x
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
0
1
2
3
4
5
6

diff1-ex3.py, diff1-ex3L.py

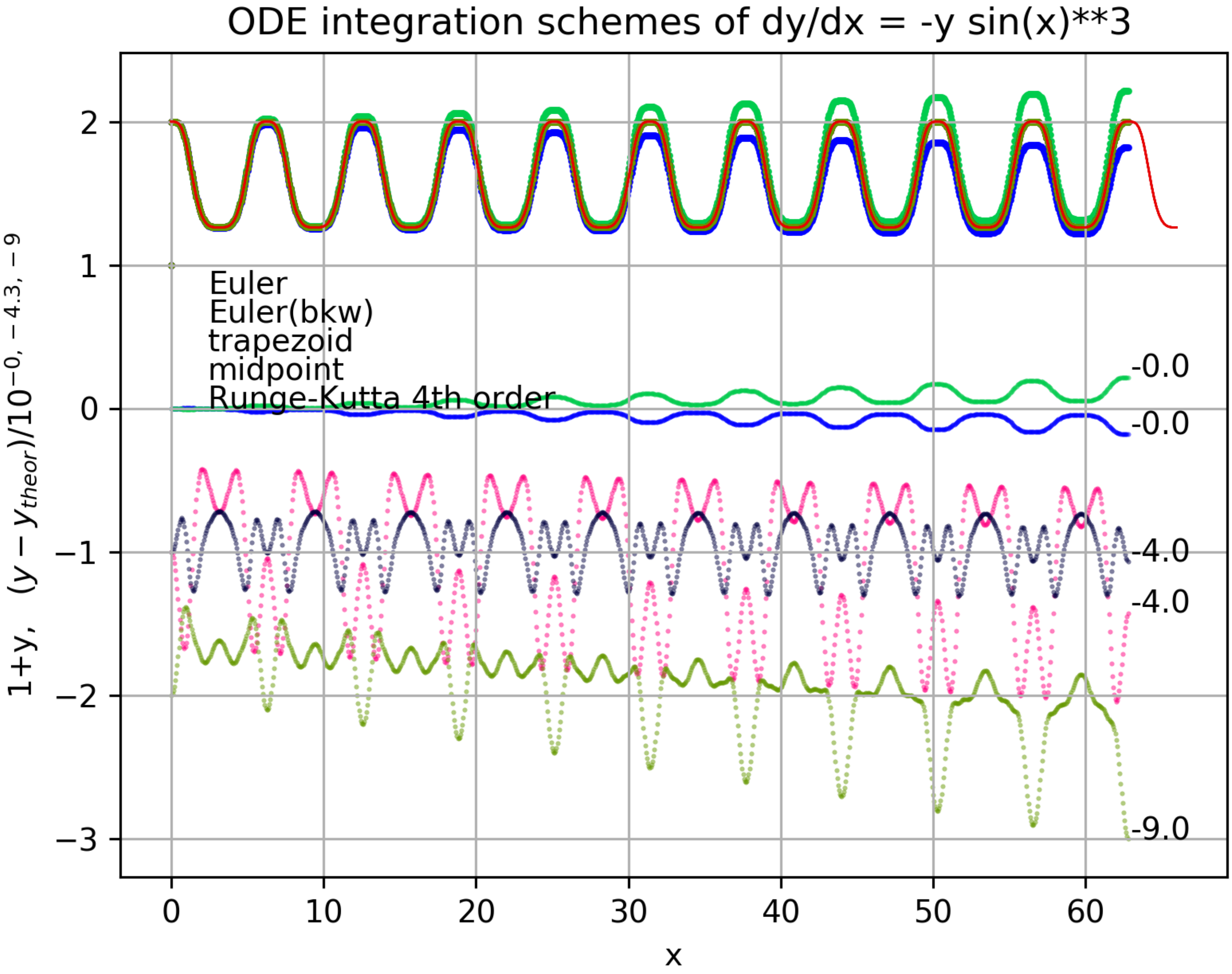


convergence of schemes



diff1-ex3.py

diff1-ex4L.py



Integration schemes

read chapter 7 of the textbook

- Midpoint trapezoid and RK4 methods
- Turner pp. 239-242

- Multistep methods – p. 245+

- Systems of equations

- Trajectories of chase

Chaotic solutions of simple regular ODEs

- Lorenz attractor – a meteorological model
  - very few variables, only 3
- The butterfly effect present
- Definition of chaos. Non-periodic behavior, extremely sensitive to perturbation.
- Orbits of Pluto and *all* other planets are chaotic too