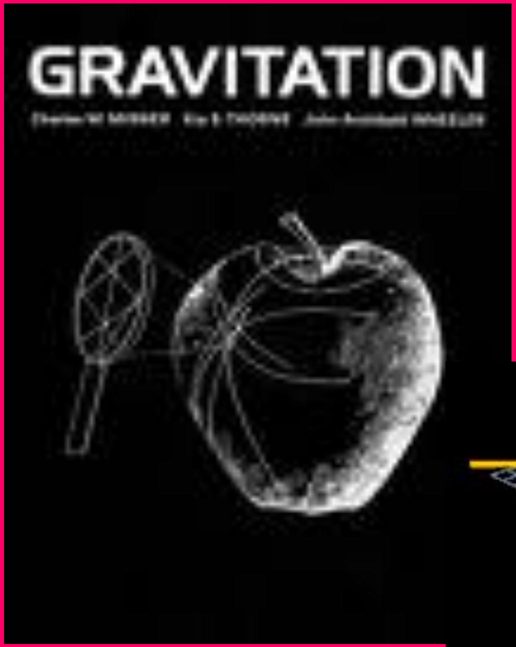
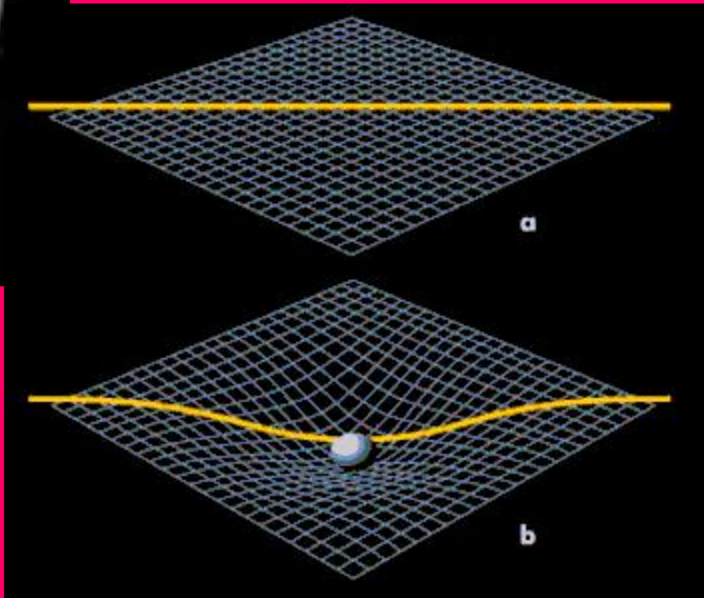
Lecture L03-04 - ASTC25

GRAVITATION

Back then





…and now

Today:

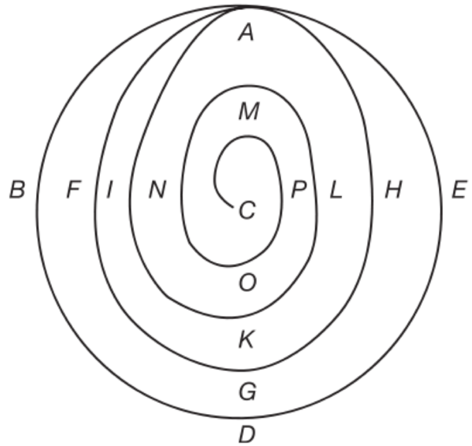
- Experiments with funnels like those that Robert Hooke did and discussed with Newton

- Centrifugal force formula (by C. Huygens) and the
- Proof that gravity falls with second power of distance using an apple and the Moon

- Arguments showing that gravity is the dominant force in the solar system, so we can neglect solar wind and small gas drag, and radiation pressure of solar radiation, when talking about planets, moons, asteroids and comets

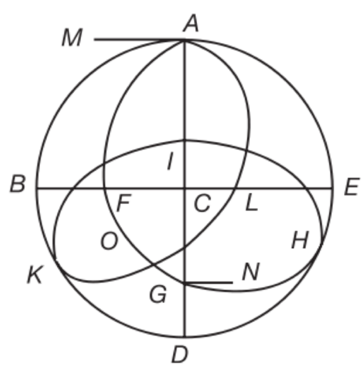
- Newton's geometric proof of Kepler's 2nd law

- The famous proof (in modern form) that 1st and 3rd Kepler's laws follow from the universal gravity force.
  The full solution of Kepler's 2-Body problem.



11. Hooke in turn argued that the body described by Newton would revolve in the ellipsoid AFGHA, unless it experienced some resistance, in which case it would descend close to the centre of the Earth

away he was from the analysis of celestial motions he would adopt seven years later in the *Principia*, but he also hinted at a much more sophisticated way of dealing with the problem according to continuous and infinitesimally small elements of gravitational force. Moreover, he implied that he could deal with a force of gravity that did not remain constant but varied from the centre outwards.



12. Newton's response, with gravity and 'centrifugal force' alternately overpowering each other

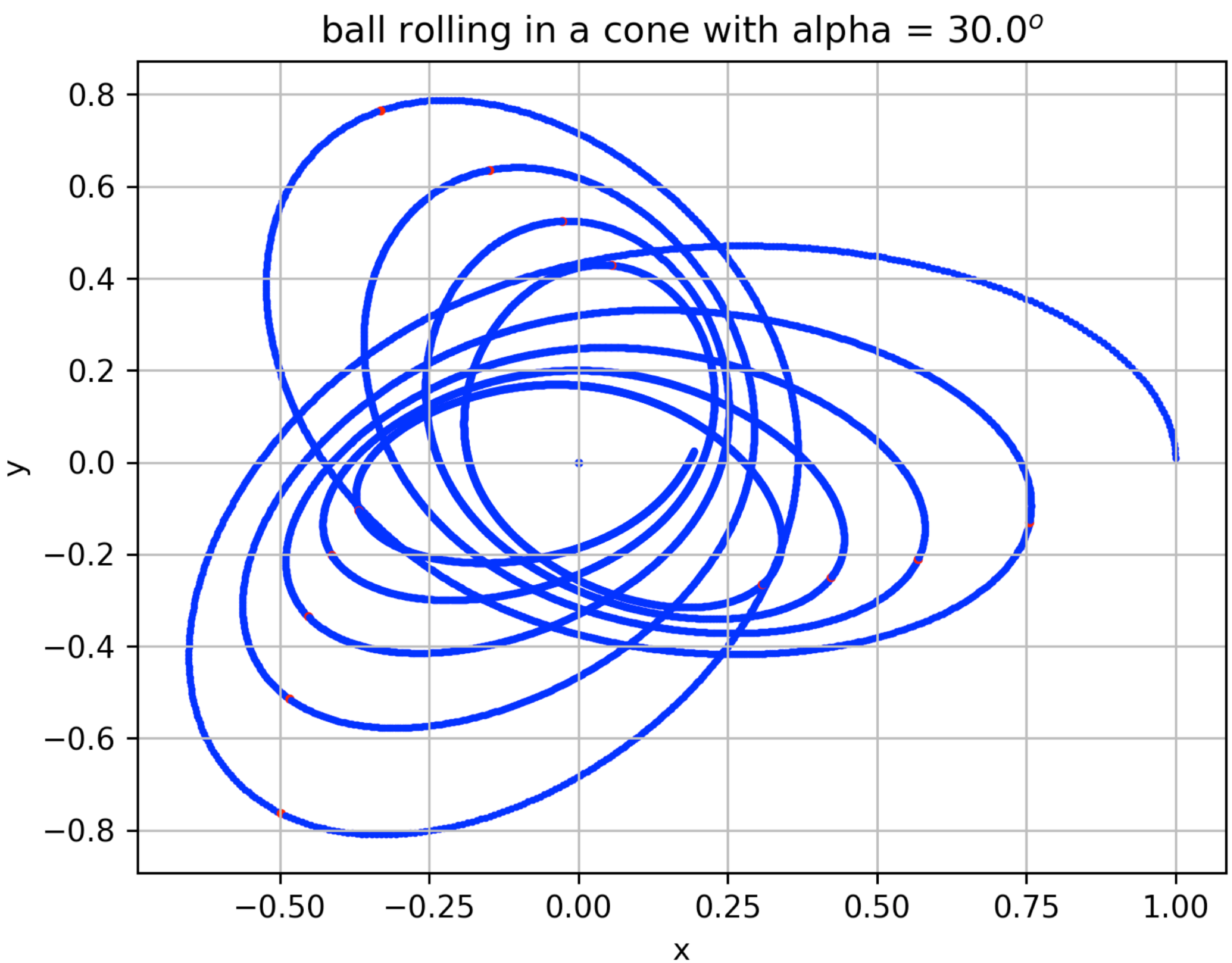
As Newton and Hooke exchanged seemingly polite letters and quarreled about trajectories, there was intense personal dislike between them..

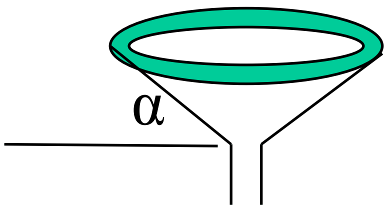
Incidentally, it shows that even in 1680s Newton did not have a full understanding of orbits and gravity. Hooke was often more right:

This suggests that Newton did not discover Universal Gravity in 1665-1666 as he claimed, but that he worked it out in response to original hypotheses of his contemporaries.

Knowing Lagrangian mechanics, today we can derive equations for a small ball rolling without sliding inside a straight cone, inclined by angle α to horizon.

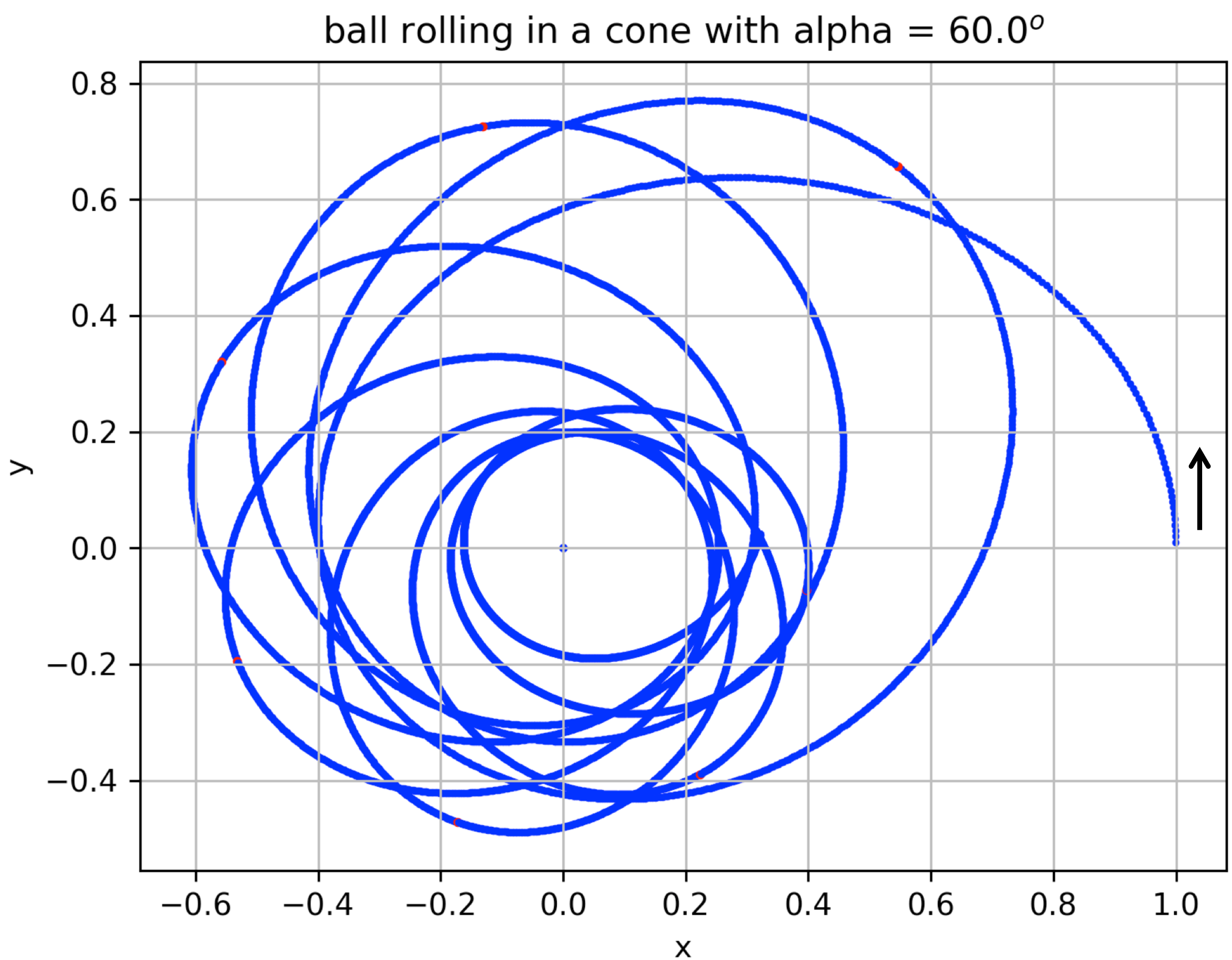
Python simulation with some friction added, top view:





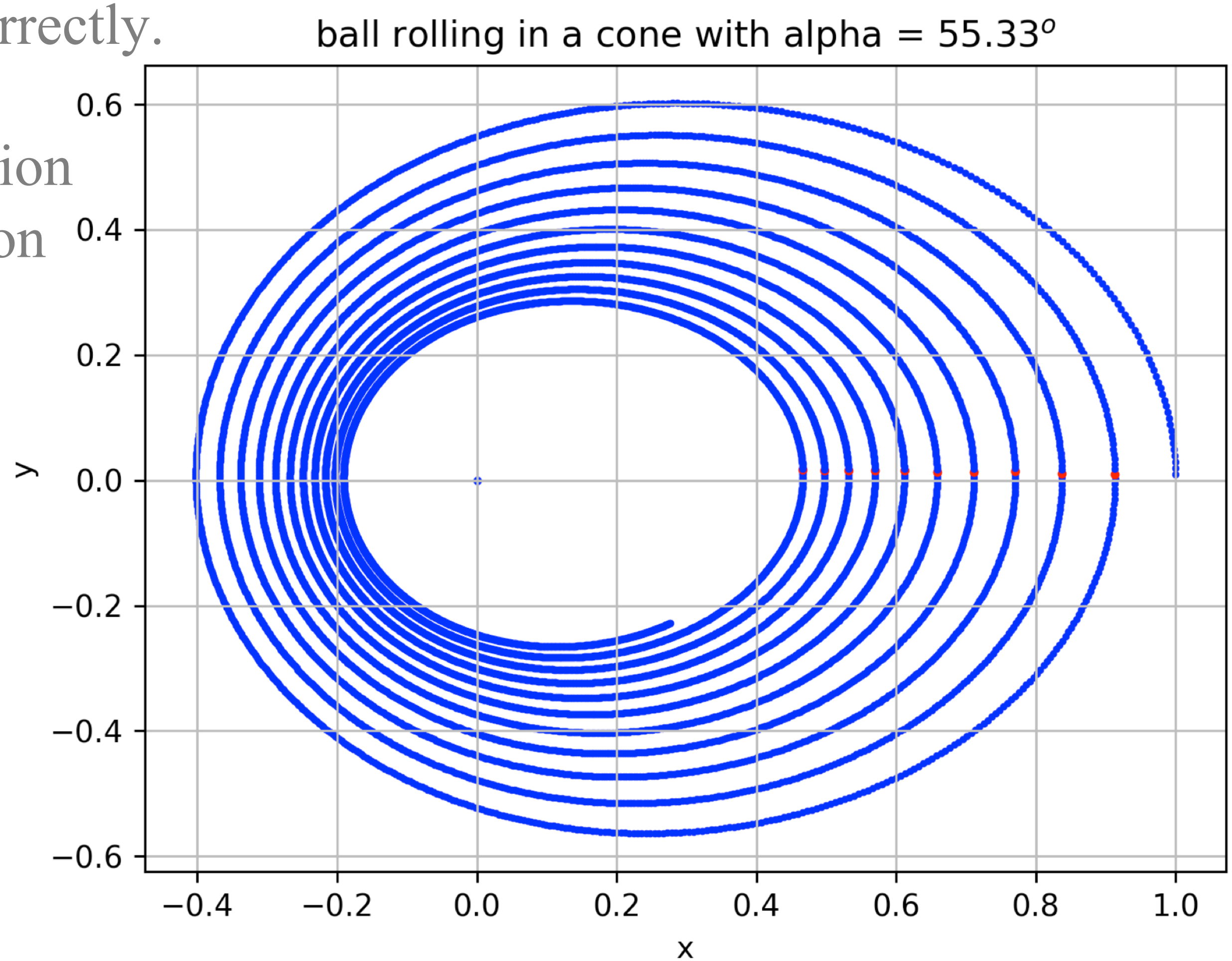
Consecutive local maxima of radius are known as apocenters. Here, they **precess,** i.e. move backwards with respect to the counterclockwise direction of motion (by about 120 degrees)

$\alpha = 60^\circ$ is typical for most commercially sold funnels: sizeable **prograde precession** is predicted (and somewhat smaller in magnitude, also observed in experiment that you can easily do & record with the slow-motion function on your smartphone.)



This angle ($\alpha = 55.33°$) in theory cancels the precession on a straight cone: $z(r) = r$, $f = -dz/dr = \tan \alpha =$ const. force law, with small friction.

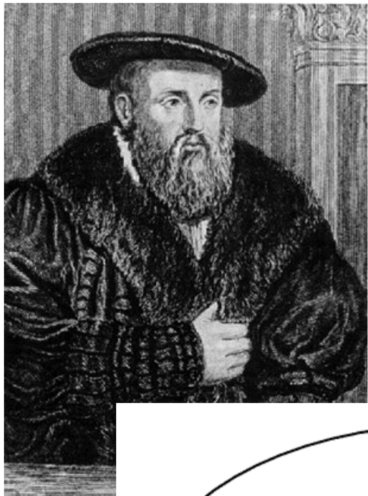
Neither Hooke nor Newton knew what angle α to chose to illustrate quasi-elliptic orbits. Better yet, they should have used a curved cone with the shape of gravity's potential, $z(r) = -1/r$, to eliminate precession and achieve better elliptical shape od orbit. It would have a slope $dz/dr = 1/r^2$, and simulate the $-1/r^2$ force of gravity correctly.



Assuring no-slip condition is difficult in low-friction cones, though.

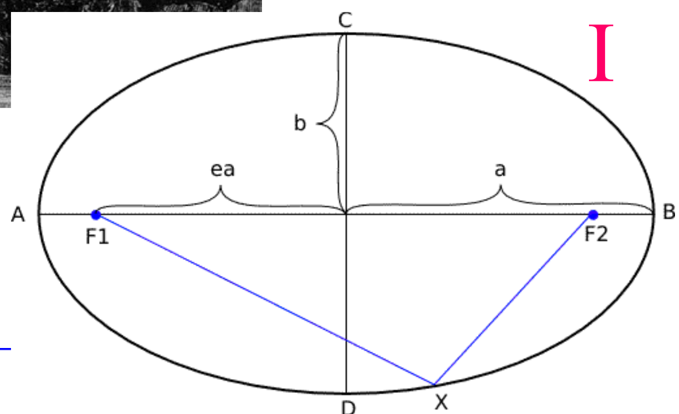
Without friction, the curve looks similar to an ellipse, but is really an oval (asymmetric egg-like) curve, differently centered than planetary orbits.

Gravity

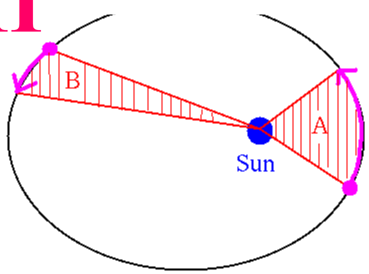


J. Kepler (1571-1630)

I



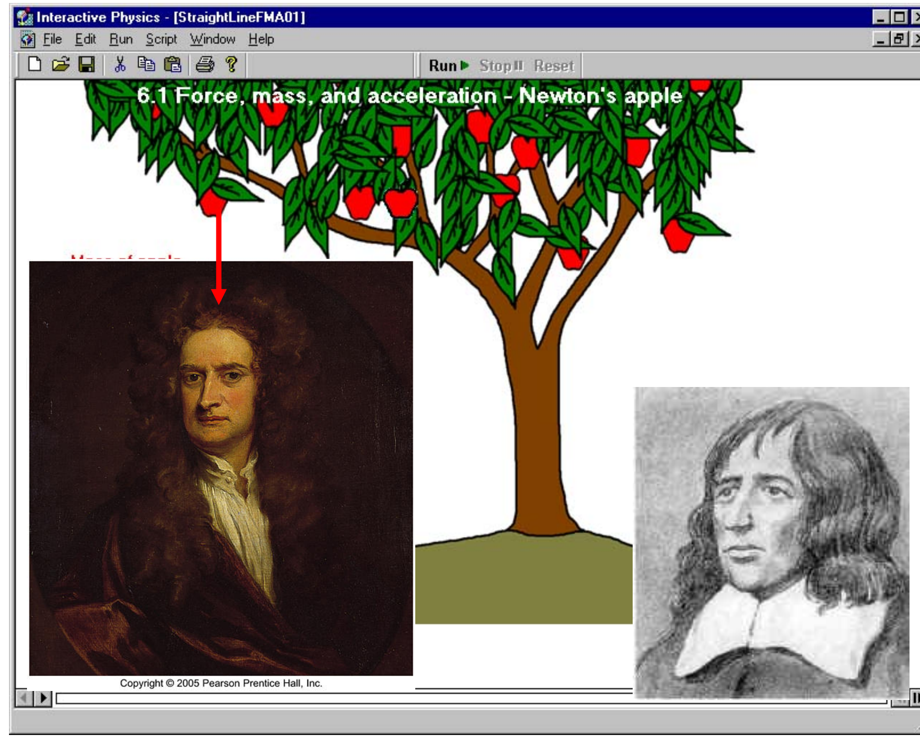
II



A = B

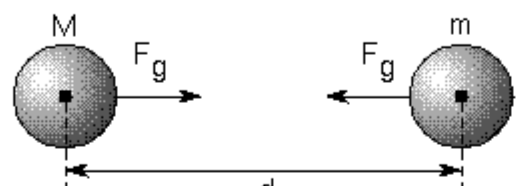
III

$$\frac{P^2}{a^3} = \text{constant}$$



I. Newton (1643-1727)

R. Hook (1635-1703)

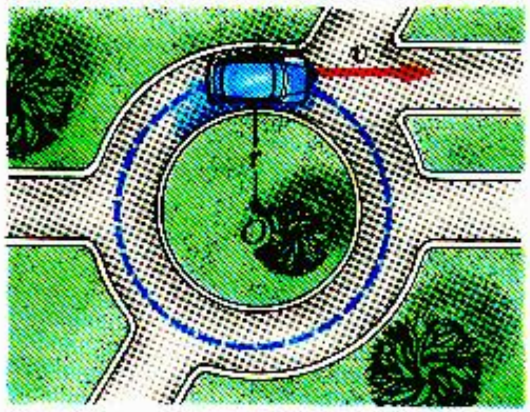


$$F_g = \frac{GMm}{d^2}$$

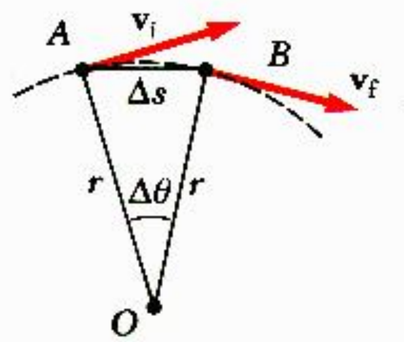
How Newton attempted a few times to confirm the
$F= -1/r^2$
law of gravity

using Huygens' formula
$a = -v^2/r$
for centripetal acceleration during uniform motion on a circle or radius r, with linear speed v = const.

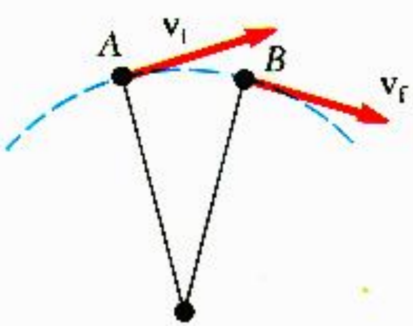
$\Delta s \cong v \cdot \Delta t = r \cdot \Delta\theta$



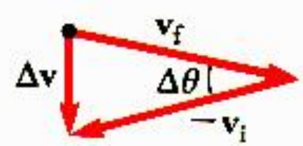
(a)



(a)



(b)



(b)

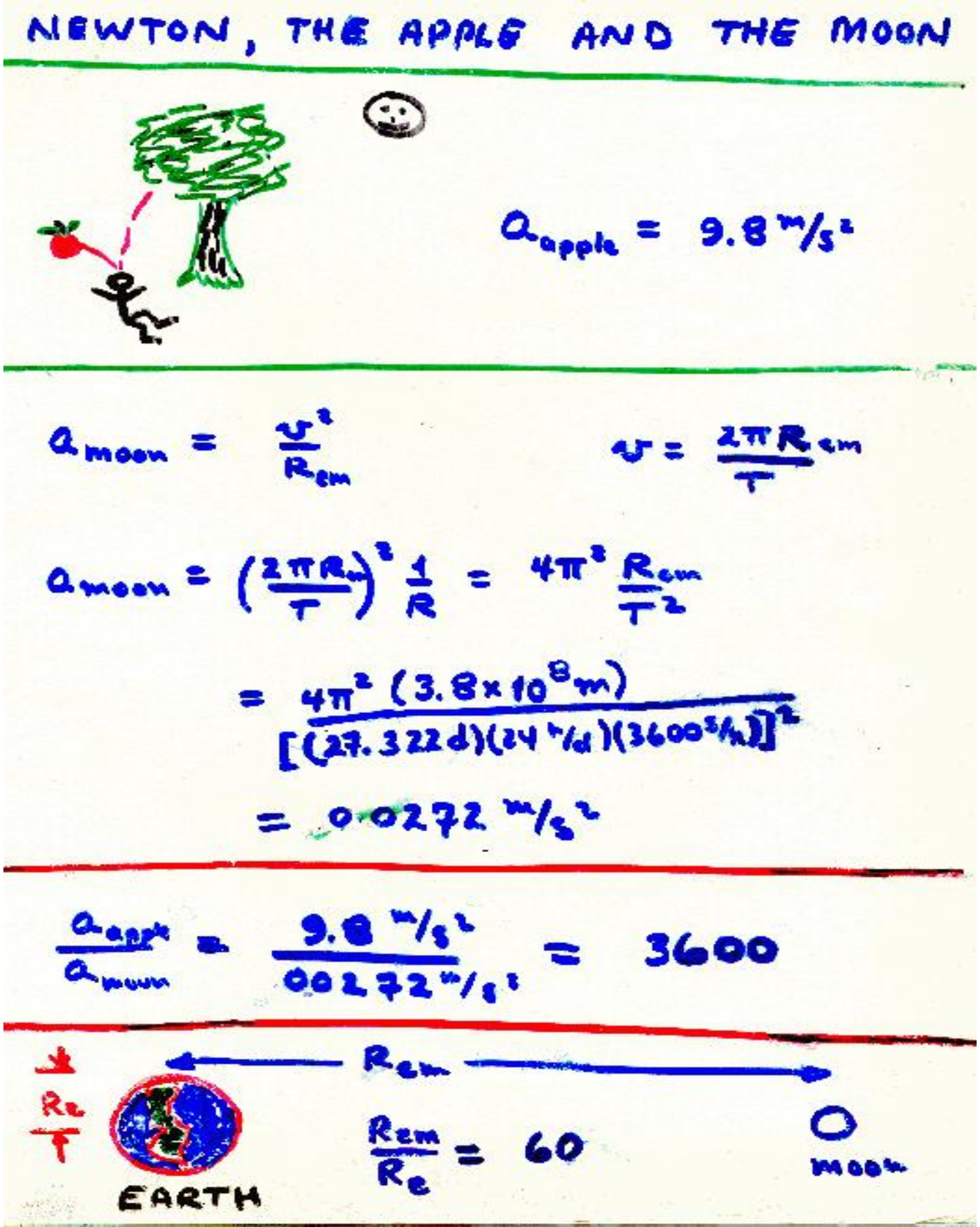
$\Delta v = a \cdot \Delta t = v \cdot \Delta\theta$

By geometry

$\Delta\theta \rightarrow 0$

$$\frac{\Delta\theta}{\Delta t} = \frac{v}{r} = \frac{a}{v}$$

$$\boxed{a = \frac{v^2}{r}}$$



Initially, Newton used inaccurate data for radius of the Earth R_E, got distance ratio >60 & was not happy with the level of agreement of $(R_{EM}/R_E)^2$ and a_{apple}/a_{moon}

We improve the calculation below.

THUS NEWTON NOTICED THAT

$$\frac{a_{apple}}{a_{moon}} = \left(\frac{R_{em}}{R_e}\right)^2 = 3600$$

AND THEREFORE DEDUCED THAT

$$F \propto a \propto \frac{1}{R^2}$$

$$\boxed{F = G\frac{mM}{R^2}}$$

Newton's Universal Law of Gravitation

If one of the masses is the earth

$$W \equiv F = G\frac{mM}{R^2} = mg$$

$$\boxed{g \equiv G\frac{M}{R^2}}$$

Guinness record (1953) of acceleration survived by someone strapped to a rocket chair:
62g for Δt=0.04s

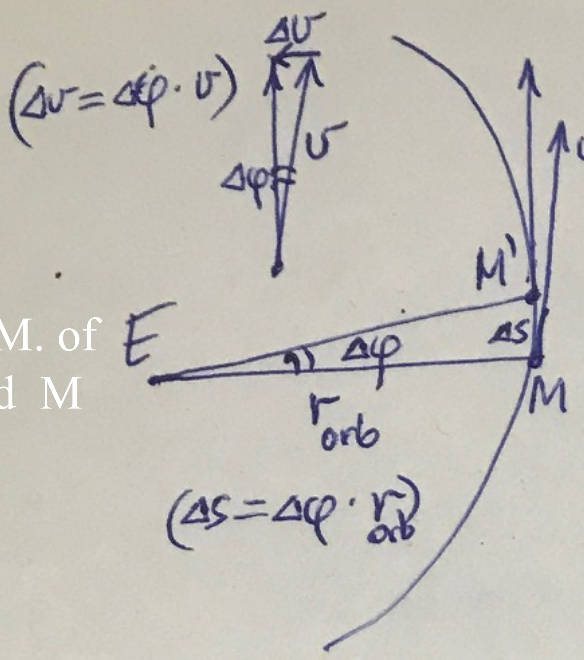
$F \sim 1/r^2$ law in times of Newton, verified for circular orbits by comparing the Moon & apple

- Let's now do the computation of Newton once again, this time with improved quality of data and treatment of center of gravity in Earth-Moon system
- We know the gravity's acceleration that applies to apples:
  $g = 9.81\ \text{m/s}^2$
- We can assume that the Moon is in a circular orbit, and that it is subject to the *same acceleration* of Earth's gravity as apple, which somehow (perhaps as $1/d^2$) falls with distance d from the center of the Earth.

- I'm going to do here the computation, using modern astronomical data (Newton knew the distance to the Moon and mainly (surprisingly) the size of Earth much less precisely, so only achieved so-so an accuracy of ~10%)

$1/r^2$ law: The apple and the Moon. A more accurate calculation.

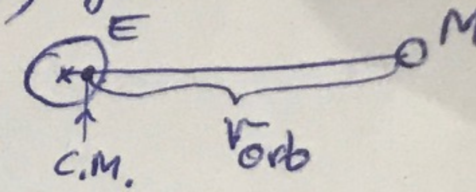
First, let's find the centripetal acceleration in a circular orbit



In time Δt the Moon moves from M to M', dist. $\hat{=} \Delta\varphi \cdot r_{orb} \equiv \Delta s$

$\Delta\varphi$ – some tiny angle

r_{orb} – orbital distance of the Moon (Moon's center) from the center of mass:



$r_{orb} = 379712$ km

$d_{EM} = 384400$ km = dist. Earth–Moon

$r_{orb} = \frac{81}{82} \cdot d_{EM}$ (mass ratio ~1:81)

Acceleration of Moon (centripetal):

$$a_r = \frac{\Delta v}{\Delta t} = \frac{\Delta\varphi \cdot v}{(\Delta s / v)} = \frac{v^2}{r_{orb}}$$

[redacted]

Another way to compute Moon's acceleration

Let's find α, is α = 2 ?

Gravitational acceleration : $a_{gr} = g\left(\frac{R}{d_{EM}}\right)^{\alpha}$

where $R = 6371$ km is Earth's radius ; $g = 9.81 \frac{m}{s^2}$

$$a_r = a_{gr} \Rightarrow \frac{v^2}{r_{orb}} = g\left(\frac{R}{d_{EM}}\right)^{\alpha}$$

What's v? Moons velocity is $\frac{2\pi r_{orb}}{P} = v$

$P = 27.3216$ days = siderial ~~per~~ month

Force balance reads:

$$\left(\frac{2\pi}{P}\right)^2 r_{orb} = g\left(\frac{R}{d_{EM}}\right)^{\alpha} \Rightarrow \alpha = \frac{\ln\left\{\left(\frac{2\pi}{P}\right)^2\left(\frac{r_{orb}}{g}\right)\right\}}{\ln(R/d_{EM})}$$

Substituting values, $\boxed{\alpha = 2.0004}$ **Yes. We got α = 2 !**

∴ same force moves apple & Moon, ~ Inverse-squared distance

From the $\pm v^2/r$ formula for centrifugal/centripetal acceleration we derive the most important relationship between speed and distance in dynamics of motion around a mass M (assuming circular, r = const., v = const. trajectory)

$$-v^2/r \;=\; -GM/r^2$$

centripetal accel. = gravitational accel., or

$$-mv^2/r \;=\; -GMm/r^2$$

(force balance for particle with small mass m)

Therefore, $\mathbf{v^2_K = GM/r}$

(Keplerian or circular speed v_K;
In motion around the Earth it used to be called 1st cosmic speed.)

The generalized formula below applies to all spherically symmetric systems, after a modification following from two theorems by Newton about the zero force inside, vs. $1/r^2$ force outside a thin uniform shell of matter.

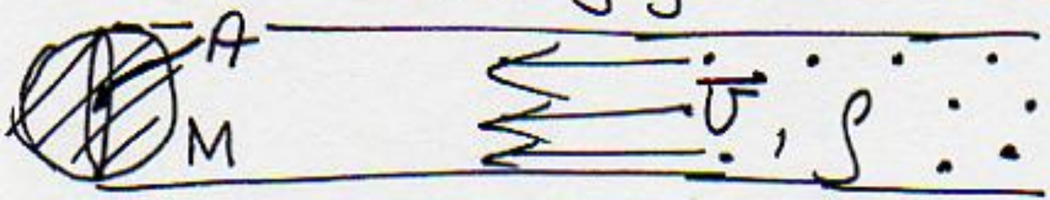
Circular speed:

$\mathbf{v_c^2 = GM(r)/r}$, where M(r) is mass inside an arbitrary radius r.

GRAVITY

- Drag forces on planets?
- 2-Body dynamics
- Roche limit and tides

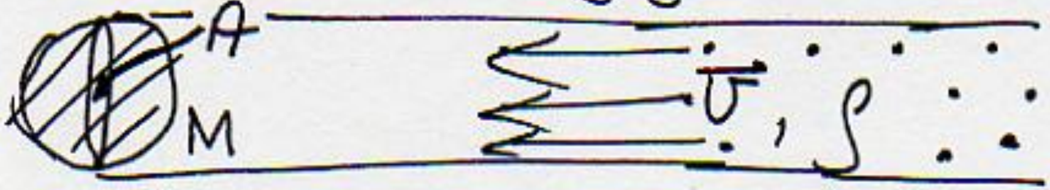
Consider drag force (acceleration) on body with cross-sectional area A and mass M, from a medium with density ρ and relative velocity $\vec{v}$



mass flux $= \frac{\Delta m}{A \cdot \Delta t} = v\rho$ units $\left[\frac{cm}{s}\right]\left[\frac{g}{cm^3}\right] =$

$\rightarrow$ intercepted $= A v \rho$ $= \left[\frac{g}{\ldots}\right]$

Consider drag force (acceleration) on body with cross-sectional area A and mass M, from a medium with density ρ and relative velocity $\vec{v}$



mass flux $= \frac{\Delta m}{A \cdot \Delta t} = v\rho$ units $\left[\frac{cm}{s}\right]\left[\frac{g}{cm^3}\right] =$

— " — intercepted $= A v \rho$ $= \left[\frac{g}{cm^2 s}\right]$

momentum per unit time intercepted $= A v\rho \cdot v$

acceleration imparted to body M: $\frac{A}{M} v^2 \rho = f$

$$\frac{A}{M} = \frac{\pi s^2}{\frac{4}{3}\pi \rho_M s^3} = \frac{3}{4 s \rho_M}$$ where $s =$ radius of body

Drag acceleration: $f = \frac{3}{4} \frac{\rho}{\rho_M} \frac{v^2}{s}$

<u>Conclusion</u>: drag force effects $\sim \frac{1}{s}$

In fact, numerical evaluation of the nondimens. ratio f/f_{gr} where $f_{gr} = \frac{GM_\odot}{r^2} = \frac{v_K^2}{r}$, $v_K = \sqrt{\frac{GM_\odot}{r}}$

$$\frac{f}{f_{gr}} = \frac{3}{4} \frac{\rho}{\rho_M} \left(\frac{v}{v_K}\right)^2 \frac{r}{s},$$

shows that only very small particle sizes s are affected in the present solar system:

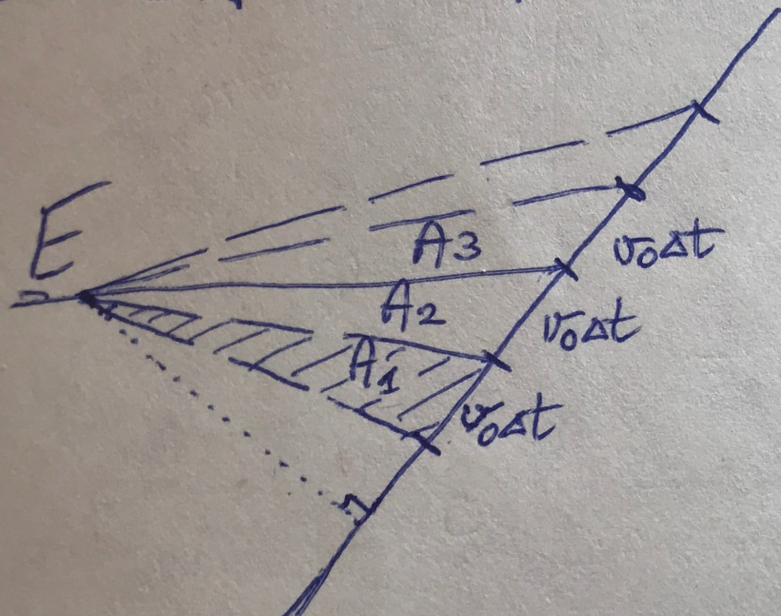
$s \leq 10^{-2}$ cm $\Rightarrow$ drag important for dust

$s \geq 10^5$ cm $= 1$ km $\Rightarrow$ drag unimportant for asteroids, planets

EXAMPLE OF GEOMETRIC PROOFS IN NEWTON'S "Principia"

Newton's proof of areal speed constancy

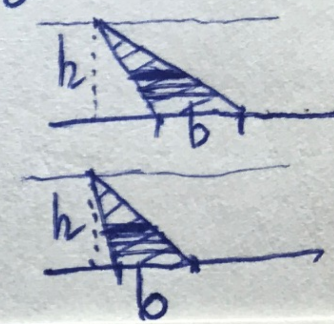
$\boxed{\text{CASE 1}}$: no forces



$A_1 = A_2 = A_3 = \ldots$

because $A = \frac{1}{2} h \cdot b = \text{const}$

if the top of the triange shifts $\parallel$ base

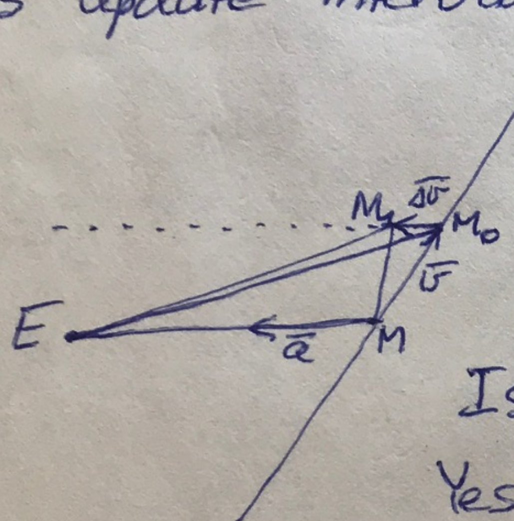


Newton's geometric proof of Kepler's 2nd law, if the force always points toward the center of attraction (centripetal force)

6

$\boxed{\text{CASE 2}}$: centripetal force

Denote acceleration as $\bar{a}$. Consider time interval Δt as update interval of velocity (later you can let $\Delta t \to 0$)



velocity gets a kick $\boldsymbol{v} \to \boldsymbol{v} + \Delta\boldsymbol{v}$ & the moon moves to M_1 not M_0

$$\overline{\Delta v} = \bar{a}\,\Delta t$$

toward E.

Is Area(EMM_0) = Area(EMM_1) ?

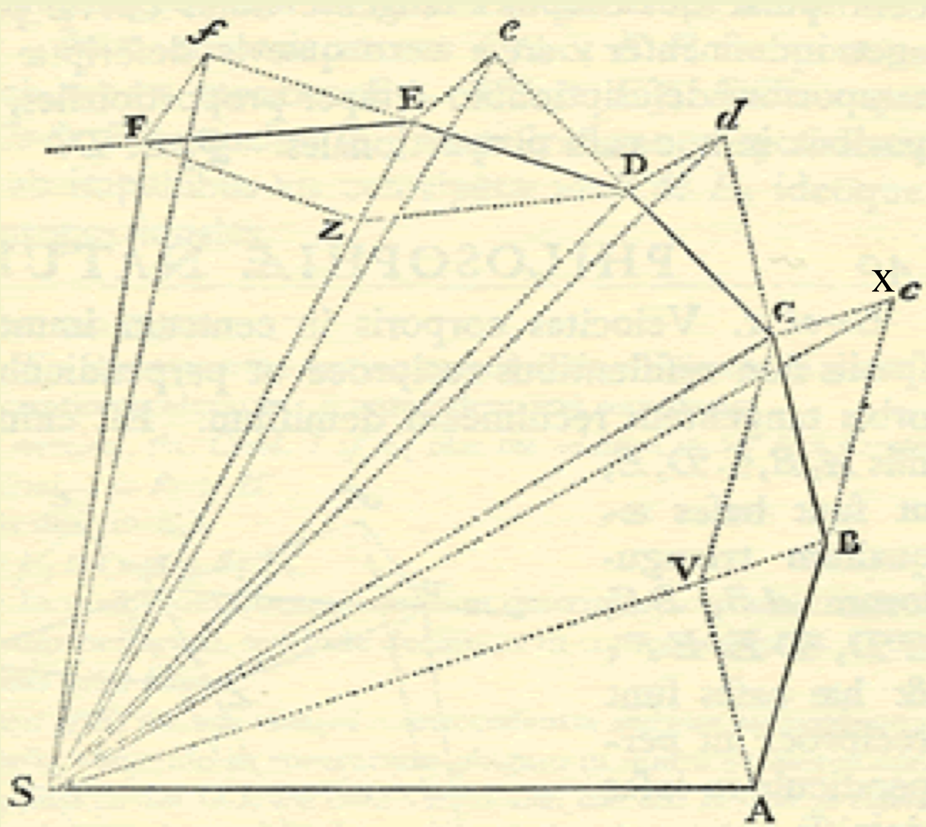
Yes, because base EM is the same and the $M_0 \to M_1$ shift is // base.

So the areal speed is constant.

trajectory consists of ∞ number of infinitesimal triangles.

Isaac Newton's geometric proof of Kepler's 2nd law, if the force points toward the center of attraction was beautifully simple

from: *Philosophiae Naturalis Principia Mathematica* (1687)



proof is based on the compound motion idea of Hooke, and elementary Euclidean geometry

In today's mechanics we prefer to talk about angular momentum $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{v}$ = const., which is a vector of length = 2 * areal speed

Proof: the cross product of two vectors, $\boldsymbol{r}$ = SA and ($\boldsymbol{v}\,\Delta t$) = AB, equals twice the area "swept" in time Δt, or area SAB.

Dividing this area by Δt gives areal speed, numerically equal to the length of $\boldsymbol{L}/2$, q.e.d.

ORBITAL MECHANICS OF PLANETARY SYSTEMS

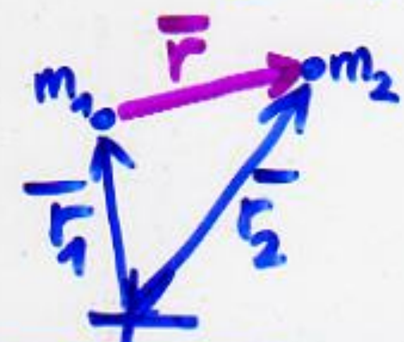
The 2-Body problem and the elliptic motion

Newton's equations of motion for 2 bodies

$$m_1 \ddot{\bar{r}}_1 = + \frac{G m_1 m_2}{r^2} \cdot \frac{\bar{r}}{r}$$

$$m_2 \ddot{\bar{r}}_2 = - \frac{G m_1 m_2}{r^2} \cdot \frac{\bar{r}}{r}$$

$$\bar{r} = \bar{r}_2 - \bar{r}_1$$



$$m_1 \ddot{\bar{r}}_1 = + \frac{G m_1 m_2}{r^2} \cdot \frac{\bar{r}}{r}$$

$$m_2 \ddot{\bar{r}}_2 = - \frac{G m_1 m_2}{r^2} \cdot \frac{\bar{r}}{r}$$

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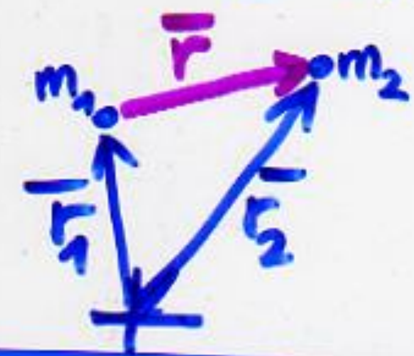


Adding them we obtain momentum conservation:

$$m_1 \ddot{\bar{r}}_1 + m_2 \ddot{\bar{r}}_2 = (\dot{\bar{p}}_1 + \dot{\bar{p}}_2) = 0$$

$\bar{r}_{CM}$:= center of mass vector

$$(m_1 + m_2)\, \bar{r}_{CM} = m_1 \bar{r}_1 + m_2 \bar{r}_2$$

$$\ddot{\bar{r}}_{CM} = 0, \quad \text{(1st Newton's law)}$$

Subtracting the equations $* \{\frac{1}{m_1}, \frac{1}{m_2}\}$:

Subtracting the equations $* \{\frac{1}{m_1}, \frac{1}{m_2}\}$:

$$\ddot{\bar r} = \ddot{\bar r}_2 - \ddot{\bar r}_1 = -\frac{Gm_1}{r^3}\bar r - \frac{Gm_2}{r^3}\bar r = -\frac{GM\bar r}{r^3}$$

where $M = m_1 + m_2$

(Reduction of 2-Body to 1-Body problem)

$$\boxed{\ddot{\bar r} = -\frac{GM}{r^3}\bar r} \Rightarrow \vec r_1 = -\frac{m_2}{M}\bar r, \quad \vec r_2 = +\frac{m_1}{M}\bar r$$

can be obtained from $\vec r(t)$, if $\vec r_{CM} = 0$.

Take vector product of $\bar r$ and □

$$\bar r \times \ddot{\bar r} = \bar r \times \left(-\frac{GM}{r^3}\bar r\right) = 0 \quad (\text{because } \bar r \times \bar r = 0)$$

But $\bar r \times \ddot{\bar r} = (\bar r \times \dot{\bar r})^{\cdot}$, because $\dot{\bar r} \times \dot{\bar r} = 0$,

so

$\vec L = \vec r \times \dot{\vec r} =$

Take vector product of $\bar{r}$ and

$$\bar{r} \times \ddot{\bar{r}} = \bar{r} \times \left(-\frac{GM}{r^3}\bar{r}\right) = 0 \quad (\text{because } \bar{r} \times \bar{r} = 0)$$

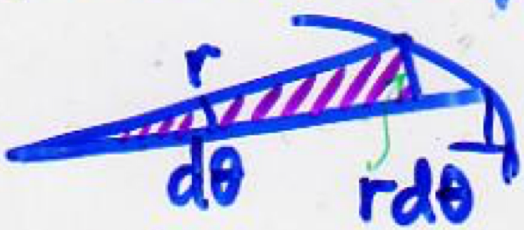
But $\bar{r} \times \ddot{\bar{r}} = (\bar{r} \times \dot{\bar{r}})^{\cdot}$, because $\dot{\bar{r}} \times \dot{\bar{r}} = 0$,

so

$$\boxed{\vec{l} = \vec{r} \times \dot{\vec{r}} = \overrightarrow{const}}$$

angular momentum conservation

II law of Kepler follows from $\vec{l} = const$



$$dA = \frac{1}{2} \cdot r \cdot r d\theta \quad \Big| : dt$$

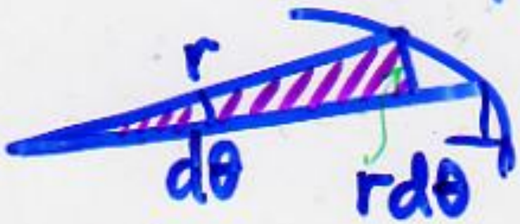
$$\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta} = \frac{1}{2} l =$$

$$= const.$$

$$l = r v_\theta = r^2 \dot{\theta}$$

$\vec{l} \perp$ orbital plane

II law of Kepler follows from $\vec{l} = \text{const}$



$$dA = \frac{1}{2} \cdot r \cdot r\,d\theta \quad \Big| : dt$$

$$\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta} = \frac{1}{2} l = \text{const.}$$

$$l = r v_\theta = r^2 \dot{\theta}$$

$\vec{l} \perp$ orbital plane

If $r(t)$ is known, $\theta(t)$ can be computed from

$$\frac{d\theta}{dt} = \frac{l}{r^2} \Rightarrow \theta = l \int^{t} \frac{dt}{[r(t)]^2}$$

---

v_θ is a very simple function of r:

$$v_\theta = \frac{l}{r}, \quad \text{or} \quad \dot{\theta} = \frac{l}{r^2}$$

Take a dot product of $\dot{\vec{r}}$ and $\square$

$$\dot{\vec{r}} \cdot \ddot{\vec{r}} = \dot{\vec{r}} \cdot \left(-\frac{GM}{r^3}\vec{r}\right)$$

or $\vec{v} \cdot \dot{\vec{v}} = -\vec{v} \cdot \vec{\nabla}\Phi$ where $\Phi = -\frac{GM}{r}$ is grav. potential

Notice $\dot{\Phi} = \cancel{\partial_t \Phi} + \vec{v} \cdot \nabla\Phi = \vec{v} \cdot \nabla\Phi$,

$$(\vec{v} \cdot \vec{v})^{\cdot} = 2\dot{\vec{v}} \cdot \vec{v},$$

hence

$$\left(\frac{v^2}{2}\right)^{\cdot} + \dot{\Phi} = 0,$$

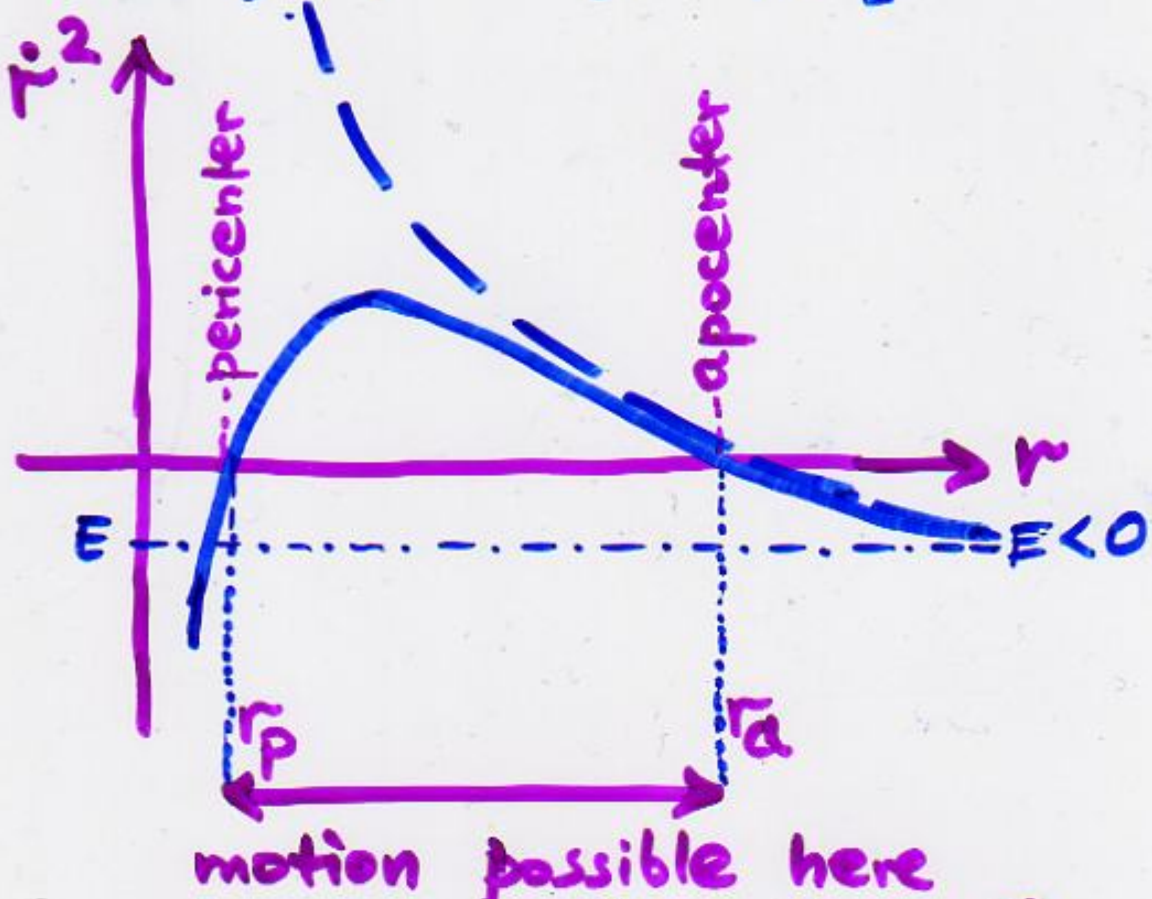
or energy

$$E = \frac{v^2}{2} + \left(-\frac{GM}{r}\right) = \text{const.}$$

En. conservation has many uses:

$$v^2 = v_r^2 + v_\theta^2 = \dot{r}^2 + \frac{\ell^2}{r^2} \Rightarrow$$

$$\Rightarrow \quad \dot{r} = \left[\left(\frac{2GM}{r} + 2E\right) - \frac{\ell^2}{r^2}\right]^{1/2}$$



One vector is so special that it had to be discovered and re-discovered a couple of times:
Laplace vector = Laplace-Runge-Lenz vector, which was discovered before Laplace(!)

Pierre-Simon, Marquis de Laplace
(1749-1827)

Finally, take the product $\boxed{\ddot{\vec{r}}} \times \vec{l}$

$$\ddot{\vec{r}} \times \vec{l} = (\dot{\vec{r}} \times \vec{l})^{\cdot} = -\frac{GM}{r^3}\,\vec{r} \times (\vec{r} \times \dot{\vec{r}})$$

Vector algebra says: $\vec{r} \times (\vec{r} \times \dot{\vec{r}}) = \underset{\overset{\|}{r\dot{r}}}{(\vec{r} \cdot \dot{\vec{r}})}\,\vec{r} - r^2 \dot{\vec{r}}.$

Therefore:

$$(\dot{\vec{r}} \times \vec{l})^{\cdot} = -\frac{GM}{r^2}\left(\dot{r}\,\vec{r} - r\,\dot{\vec{r}}\right)$$

Notice $\left(\frac{\vec{r}}{r}\right)^{\cdot} = \dot{\hat{r}} = \frac{\dot{\vec{r}}}{r} - \frac{\dot{r}}{r^2}\vec{r} = \frac{r\dot{\vec{r}} - \dot{r}\vec{r}}{r^2}$,

so

$$\left(\dot{\vec{r}} \times \vec{l} - \frac{GM\vec{r}}{r}\right)^{\cdot} = 0$$

$$\vec{e} := \frac{\dot{\vec{r}} \times \vec{l}}{GM} - \frac{\vec{r}}{r} = \overrightarrow{\text{const}}.$$

Laplace–Runge–Lenz vector conserv.

$$\vec{e} := \frac{\dot{\vec{r}} \times \vec{l}}{GM} - \frac{\vec{r}}{r} = \overrightarrow{\text{const.}}$$

Laplace-Runge-Lenz vector conserv.

Properties and consequences:

$$\vec{e} \cdot \vec{l} = \underbrace{(\dot{\vec{r}} \times \vec{l}) \cdot \vec{l}}_{\text{same}} - \frac{GM}{r} \underbrace{\vec{r} \cdot (\vec{r} \times \dot{\vec{r}})}_{\text{same}} = 0.$$

Laplace vector is in the orbital plane, and $\vec{e} \perp \vec{l}$. Let's call the angle between $\vec{e}$ and $\vec{r}$ θ, then

$$\vec{e} \cdot \vec{r} = e r \cos\theta = \frac{\vec{r} \cdot (\dot{\vec{r}} \times \vec{l})}{GM} - r$$

But $\vec{r} \cdot (\dot{\vec{r}} \times \vec{l}) = \vec{l} \cdot (\vec{r} \times \dot{\vec{r}}) = \vec{l} \cdot \vec{l} = l^2$;

so

$$r(e \cos\theta + 1) = \frac{l^2}{GM}$$

If we define a new constant a as :

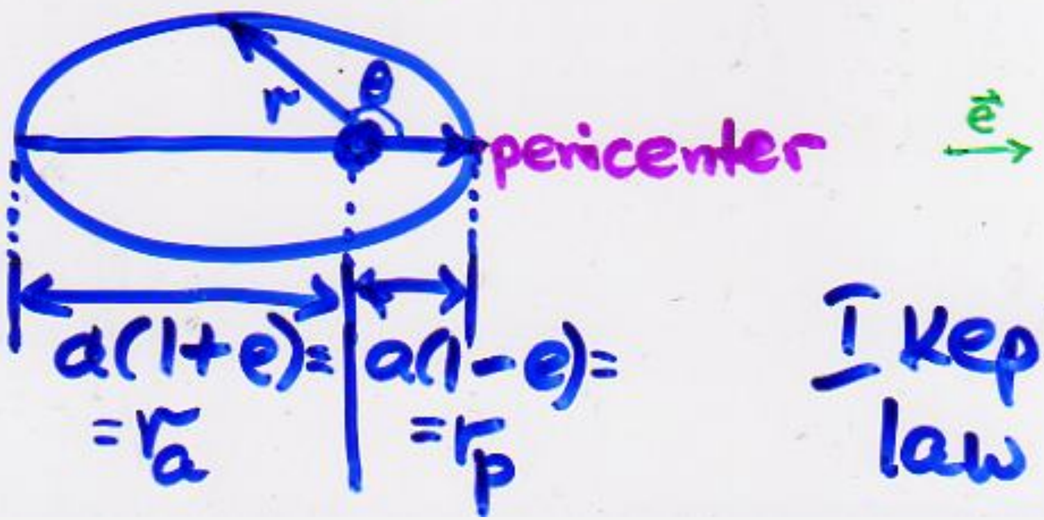
$$a(1-e^2) = \frac{\ell^2}{GM}$$ ..so $\boxed{l = \sqrt{GMa(1-e^2)}}$

the $\vec{e}\cdot\vec{r}$ formula becomes an equation of

$$\boxed{r = \frac{a(1-e^2)}{1+e\cos\theta}}$$

ellipse

with semi-major axis a, and eccentricity e



I Kepler's law

eccentricity	orbit	energy
$e = 0$	circular	$E < 0$
$0 < e < 1$	elliptic	$E < 0$
$e = 1$	parabolic	$E = 0$
$e > 1$	hyperbolic	$E > 0$

In general $\boxed{E = -\frac{GM}{2a}}$, because, e.g.

at pericenter $E = \frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\theta}^2 - \frac{GM}{r} =$

$$= 0 + \frac{l^2}{2r_p^2} - \frac{GM}{r_p} = -\frac{GM}{2a} \quad \square$$

Exercises:

(i) Derive the shape of the orbit in case e = 1 explicitly, in Cartesian coordinates

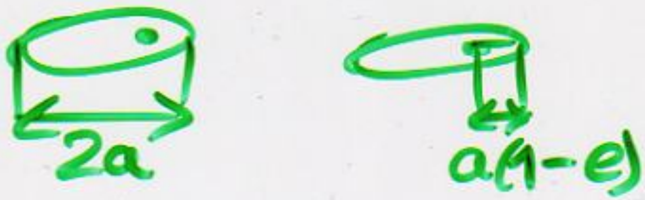
(ii) Verify the proof of $E = \frac{-GM}{2a}$.

Notice: *we often designate spec. ang. momentum as L instead of* ℓ

$\{E, \ell\} \longleftrightarrow \{a, e\}$

$$\begin{cases} E = -\dfrac{GM}{2a} \\ \ell^2 = GMa(1-e^2) \end{cases}$$

are the two alternative sets of parameters describing the size (a) and the shape (e) of the ellipse



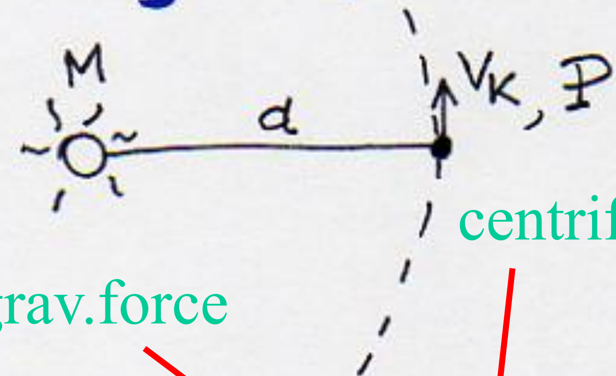
other orbital elements describe the orientation in space (3 angles) and a time constant.

Together there are 6 orbital elements:

$$a, e, i, \omega, \Omega, t_0$$

$$\frac{v^2}{2} = \frac{\dot{r}^2}{2} + \frac{GMa(1-e^2)}{r^2} = GM\left(\frac{1}{r} - \frac{1}{2a}\right)$$

III Kepler's law
(easy to prove in $e=0$ case)



$$\ddot{r} = f_r + \frac{v_\theta^2}{r} =: f_r + \frac{v_K^2}{r}$$

$\ddot{r} = 0$ because $r = a = \text{const.}$

$$f_r = -\frac{GM}{a^2}$$

centrif. force

grav.force

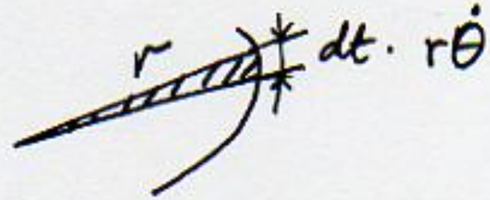
$$\Rightarrow \quad \frac{GM}{a^2} = \frac{v_K^2}{a}, \quad \boxed{v_K^2 = \frac{GM}{a}},$$

(Make sure <u>you</u> know how to derive Keplerian speed !)

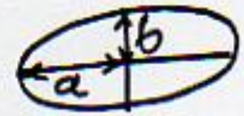
$$P = \frac{2\pi a}{v_K} = 2\pi\sqrt{\frac{a^3}{GM}} \quad \Rightarrow \quad \boxed{\frac{P^2}{a^3} = \frac{(2\pi)^2}{GM}} \quad \square$$

III Kepler's law (via II law)

$$\frac{dA}{dt} = \frac{1}{2} r \cdot r\dot{\theta} = \frac{1}{2}\ell$$



$b^2 = a^2(1-e^2)$
e = eccentricity



$$dA = \frac{1}{2}\ell\, dt, \quad \int dA = \frac{1}{2}\int \ell\, dt,$$

$$A = \frac{1}{2}\ell P$$

Geometry of ellipse $\Rightarrow$ $A = \pi ab = \pi a^2(1-e^2)^{\frac{1}{2}}$

$$\ell = \sqrt{GMa(1-e^2)}$$

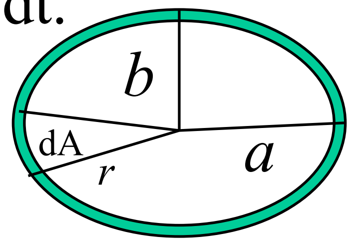
Hence,

$$\pi a^2 = \frac{P}{2}\sqrt{GMa}$$

$$\frac{P^2}{a^3} = \frac{(2\pi)^2}{GM} \quad \square$$

GENERAL PROOF OF 3rd KEPLER's LAW again, nicely typewritten:

In non-circular case, the 3rd Kepler's law follows from the 2nd Kepler's law: areal speed $dA/dt = \text{const.}$, dA = area swept in time dt.



$(b/a)^2 = 1 - e^2$

Area of the ellipse equals $A = \pi ab$, but at the same time $A = \oint (dA/dt)\, dt = (dA/dt) \oint dt = (dA/dt)\, P$. Here

P = orbital period

a = semi-major axis = half of the bigger axis

b = semi-minor axis = half of the smaller axis of ellipse.

$$dA/dt = (\tfrac{1}{2} r^2 d\theta)/dt = \tfrac{1}{2} r\, v_\theta = \tfrac{1}{2} L = \text{const.} \qquad L = l = |\mathbf{r} \times \mathbf{v}|$$

Of the two components of velocity: $v_\theta = r\, d\theta/dt$, and $v_r = dr/dt$, only the first sweeps the area $dA \sim dt$, the second $dA_2 \sim (dt)^2$.

In the limit of $dt \to 0$, only dA results in a finite area.

$$P = \pi ab \,/\, (\tfrac{1}{2} L)$$

But $L = [GM\, a(1-e^2)]^{1/2}$ from the previous derivations, and

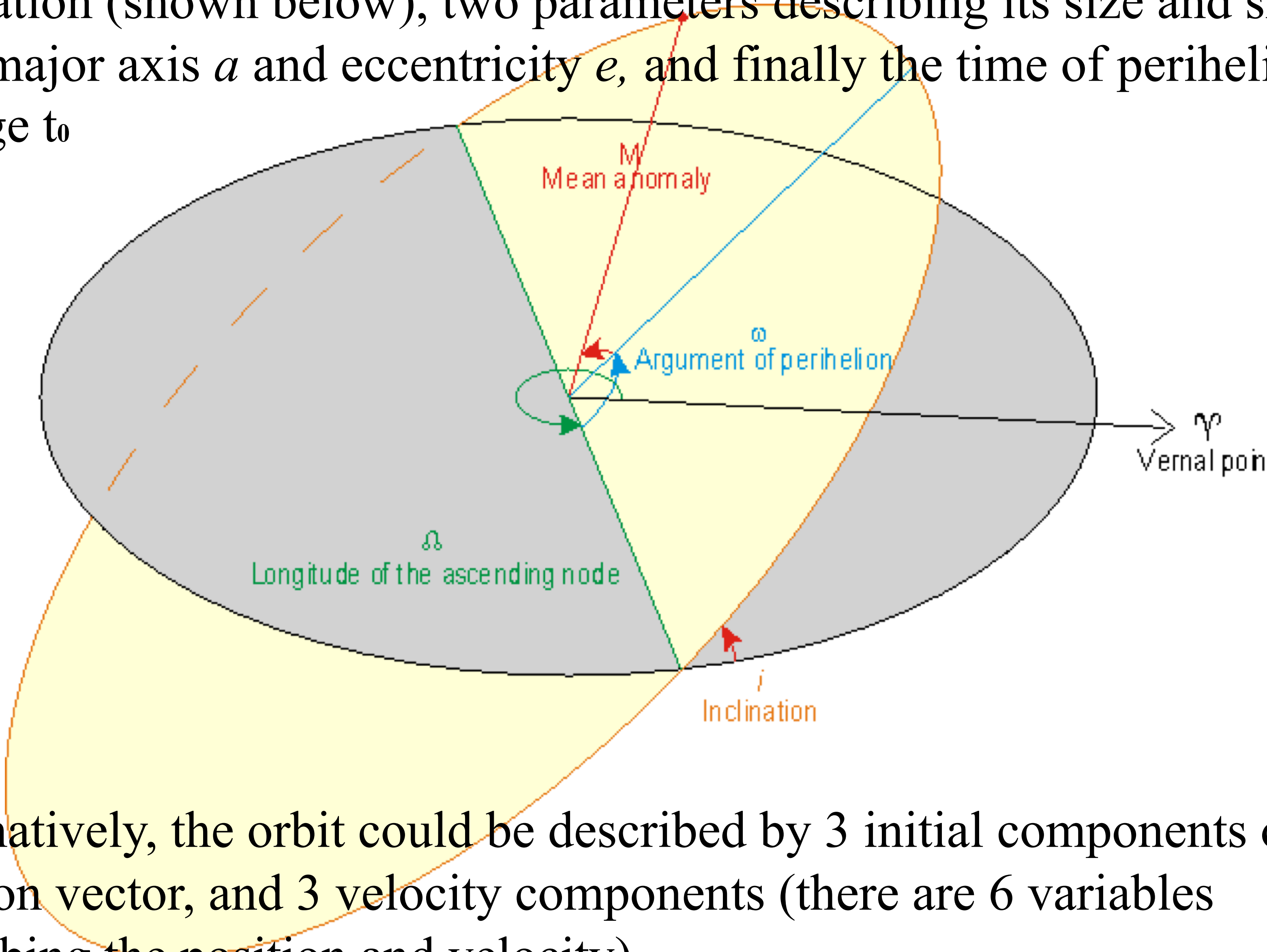
$b = a(1-e^2)^{1/2}$ from the equation of ellipse, therefore

$$P = 2\pi\, (a^3/GM)^{1/2} \qquad \text{(3rd Kepler's law).}$$

Symbols Ω or n denote the average angular speed, or 'mean motion'

$$\Omega = n = 2\pi/P = (GM/a^3)^{1/2}$$

The full set of orbital elements (constants describing a Keplerian orbit) includes the two omegas and the inclination angle *I,* describing orbit's orientation (shown below), two parameters describing its size and shape: semi-major axis *a* and eccentricity *e,* and finally the time of perihelion passage t_0



Alternatively, the orbit could be described by 3 initial components of position vector, and 3 velocity components (there are 6 variables describing the position and velocity).

ADDITIONAL LITERATURE

You can find interesting information about the history of celestial in many places on internet.

A great place to look is Lissauer textbook “Fundamentals of Planetary Science”, chapter 2 on Orbital Mechanics.

If want a different formulation, see the annotated chapter 2 from the Ostlie+Carroll textbook linked to our home page at
https://planets.utsc.utoronto.ca/~pawel/ASTC25