The Role of Non-Linearity in Soliton Solutions and Applications

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Abstract

This paper studies soliton solutions of nonlinear differential equations. Soliton solutions take the form of stable local waves capable of retaining their properties even after interactions. This paper introduces the Korteweg-de Vries equation and the non-linear Schrödinger equation. An outline of different solution methods, such as the inverse scattering transform, is given and the key results are summarized. The characteristics of the resultant solutions are analyzed numerically by observing single and double soliton solutions. Propagation and interaction are also analyzed numerically. Finally, an overview of various soliton applications is provided to highlight the significance of the non-linearity aspect. The study of such equations proves to be of great theoretical and practical interest.

Contents

1 Introduction

Solitons, or solitary waves, were a phenomenon observed in canals in the 18th century by John Scott Russell, who recorded his discovery that the movement of water presented a large, round, smooth, solitary pile of water. The pile of water continued to move without appreciably changing shape or slowing down and finally disappeared several miles away. In later water tank experiments, Russell noticed that the 'water piles' or waves were stable and could travel great distances. They never merged, but the more giant waves overtook the smaller ones. Russell's results were different from existing wave conclusions. Scientists debated the phenomenon discovered by Russell until Korteweg and de Vries proposed the famous KdV equation in 1895 which bears their name.

'Zabusky and Kruskal conducted a numerical analysis of the KdV equation in 1965 and established the meaning of solitary waves. They observed that solitary waves maintained their shape and speed after collisions and used the term "soliton" to describe them (Solomon Manukure & Timesha Booker, 2021)'. Solitons are specific solutions of a class of nonlinear partial differential equations with special elementary solutions. These solutions, called solitons, are stable and have the form of local waves that retain their properties even after the interaction

2 Equations with Soliton Solutions

2.1 From Linear to Non-Linear

The one-dimensional wave equation $u_{tt} - c^2 u_{xx} = 0$ is the most fundamental linear partial differential equation (PDE). The solution is given by $u(x,t) = F(x - ct) + G(x + ct)$. The general solution follows the superposition principle, which is the sum of a right-traveling function F and a left-traveling function G, which depend on the initial state and boundary conditions. A function that satisfies the solution, such as $u(x,t) = \operatorname{sech}(x - ct)$, can be verified by substituting into the wave equation. However, obtaining nonlinear differential equations is much more complex. Nonlinear equations do not satisfy the superposition theorem. Consider a fundamental equation of fluids(Inviscid Burgers' equation): $u_t + uu_x = 0$

A possible way is to use the geometric method to solve the PDE by studying the characteristic curves given by the ordinary differential equation (ODE) $dx/dt = u(x,t)$. However, since the PDE is nonlinear, the characteristics depend on $u(x,t)$ itself. Each solution will have a different set of characteristics. There is no universal solution technique for such differential equations, and so each individual equation needs to be studied separately.

2.2 KdV Equation

The Korteweg-de Vries (KdV) equation is a general model for many different physical situations, an example of which is the formula derived from the shallow water wave model in hydrodynamics. It was the first formula to analytically confirm the existence of soliton solutions. This made it the most famous partial differential equation in the study of soliton physics:

$$u_t + 6uu_x + u_{xxx} \tag{1}$$

2.2.1 Solving by Conversion to an ODE

One way to solve the Kdv equation is by converting the PDE to and ODE and integrating it (Strauss, W. A. 2008). Begin with an ODE of the form

$$-cf' + 6ff' + f''' = 0$$

Integration is done twice, using the boundary condition that the wave is solitary as $x \to \infty$, yielding $(f')^2 = f^2(c - 2f)$. Integrating one final time, and using the initial condition that $x = x_0$ at $t = 0$, the solution is:

$$u(x, t) = \frac{c}{2} \operatorname{sech}^2 \left[\frac{\sqrt{c}}{2}(x - x_0 - ct) \right] \tag{2}$$

2.2.2 Solving using the Inverse Scattering Transform

Another way to solve this class of PDEs is using the inverse scattering transform (IST), which was developed in 1967 by Gardner, Greene, Kruskal, and Miura (Abel, S. 2009). The IST is a method based on highly nonlinear equations that have a close and complex relationship with linear equations, such as the linear Schrödinger equation:

$$\Psi_{xx} + u\Psi = \lambda\Psi$$

This method gets $u(x, t)$ from the initial value $u(x, 0)$ with three steps: scattering, time evolution, and inverse scattering. For the linear Schrödinger equation, begin with initial data $u(x, 0)$, which plays the role of the potential. It takes into account the reflection and transmission coefficients of the equation as the initial scattering data. After time evolution, the potential $u(x, t)$ at time t is reconstructed by the inverse scattering. That is, given the set of scattering data, it can reconstruct the potential that the particles have scattered off.

Consider the $V(x) = \alpha \operatorname{sech}^2(x)$ in the linear Schrödinger equation.

$$\Psi_{xx} - k^2\Psi = V\Psi$$

The equation's solution is associated with Legendre polynomials (Abel, S. 2009). Then consider $k^2 < 0$, this gives $\alpha = N(N + 1)$, which suggests an N-soliton solution with an initial value $u(x,0) = N(N+1)\,\mathrm{sech}^2(x)$. For example $N = 2$ suggests $u(x,0) = 6\,\mathrm{sech}^2(x)$ as an initial value. Another example will consider that N = 1. In this case, after IST, $2\,\mathrm{sech}(x-4t)$ will be the solution.

For more information on the inverse scattering transform and an overview of the underlying steps, see Appendix Listing 1.

2.3 Non-Linear Schrödinger Equation

Another important equation that has spatially localized solutions is the non-linear Schrödinger equation (NLS). It has many applications, especially in fiber optics, where the soliton's stability can be used to transmit data over long distances. The equation has the form:

$$i\frac{\partial \Psi}{\partial t} + P\frac{\partial^2 \Psi}{\partial x^2} + Q|\Psi|^2\Psi = 0 \tag{3}$$

Where the coefficients P and Q depend on the particular problem which is being studied. This equation is analogous to the Schrödinger equation only if the P is positive. However, the equation can be transformed to make sure that P is positive because if $P < 0$, we can change the sign of the equation and then restore the positive sign in front of the time derivative by taking the complex conjugate. This leads to an equation for Ψ^* in which the coefficient of $\frac{\partial^2 \Psi^*}{\partial x^2}$ is positive. Therefore, P can be assumed to be positive without any restrictions.

The potential term is equal to $-Q|\Psi|^2$ which is the nonlinear term in the NLS equation. It turns out that if Q is positive, the solutions for Ψ are localized. Furthermore, it generates a potential well for $-Q|\Psi|^2$, which is a necessary condition for the NLS equation to have a spatially localized solution.

2.3.1 Solution Methodology

To solve the nonlinear differential equation, we look for solutions of the form

$$\Psi = \Phi(x,t)e^{i\theta(x,t)} \tag{4}$$

Where Φ and θ are real functions. The full derivation of the NLS equation is given in Appendix Listing 2. It is quite lengthy, so for now the final solution is presented:

$$\Psi(x,t) = \Phi_0\,\mathrm{sech}\left[\sqrt{\frac{Q}{2P}}\Phi_0(x-u_e t)\right] e^{i\frac{u_e}{2P}(x-u_p t)} \tag{5}$$

which can also be written in the form

$$\Psi(x,t) = \Phi_0\,\mathrm{sech}\left[\sqrt{\frac{x-u_e t}{L_e}}\right] e^{i(\kappa x - \mu t)} \tag{6}$$

where L_e, κ, μ are defined as follows:

$$L_e = \frac{1}{\Phi_0}\sqrt{\frac{2P}{Q}} \qquad \kappa = \frac{u_e}{2P} \qquad \mu = \frac{u_e u_p}{2P} = \kappa u_p$$

This shows that the solution to the NLS equation is a wave packet with a width of L_e, which is inversely proportional to the amplitude Φ_0. This is similar to the solution of the KdV equation, which also has the hyperbolic secant function. Thus, we recover a soliton-like solution which arises from the nonlinearity of the NLS equation.

3 Numerical Analysis

3.1 Single Soliton Solution to the KdV Equation

Since the single and two soliton solutions are in one-dimensional space, we can plot the result to obtain the behaviour. Consider an initial value of $x0 = 0$ when $t = 0$, and the speed of the wave is 4. We can obtain a soliton solution such that $u(x,t) = 2\,\mathrm{sech}^2(x - 4t)$, in this case, $c = 4$.

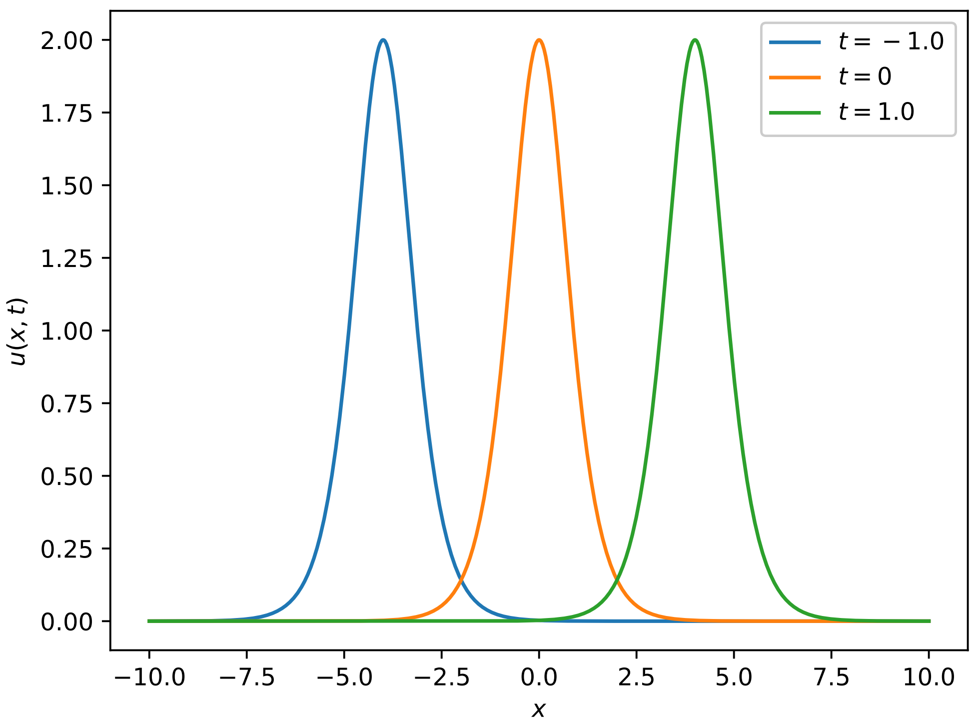

Figure 1: A plot of the single soliton solution $u(x,t) = 2\,\mathrm{sech}^2(x - 4t)$ at 3 different timestamps.

The solitary form of the hyperbolic secant function propagates at a constant speed and in a single direction without changing shape. If c is large, the wave is tall, thin, and fast. Conversely, if c is small. it is short, fat, and slow. Specifically, for the KdV equation or other similar equations, the soliton wave is the single soliton solution of the equation. When there are multiple solitons in a solution, we call the solution a soliton solution. In most cases, the N-soliton solution of the KdV equation has N-independent peaks, and each individual peak is called a soliton. When a soliton is infinitely separated from any other soliton, the soliton becomes solitary.

3.2 Two Soliton Solution to the KdV Equation

Since the Kdv equation is nonlinear, the superposition solution is not a soliton. For two-soliton solutions, interactions between multi-solitons are more complex because they involve interactions between different superposition solutions. An example is the two soliton solution:

$$u(x,t) = \frac{12[3 + 4\cosh(2x - 8t) + \cosh(4x - 64t)]}{[3\cosh(x - 28t) + \cosh(3x - 36t)]^2} \tag{7}$$

The basic theory of the double soliton solution gives the initial state: $u(x,0) = N(N+1)\,\text{sech}^2(x)$, when $N = 2, N(N+1) = 6$. Since the soliton solution is a special solution to the formula, it can be verified by substituting $u(x,0)$ into the formula. As shown earlier, $u(x,0) = 6\,\text{sech}^2(x)$ is a soliton solution. No oscillations are expected at infinity for such values, and the potential is then reflection-less.

The solution in Figure 2 used $t \in (-\infty, \infty)$. The taller wave catches the shorter one and they merge to form a single wave at $t = 0$. After which the taller wave reappears to the right and moves away from the shorter wave as t increases. The interaction of waves is non-linear. The taller wave moved forward, and the shorter one moved backward relative to the positions they would have reached if the interaction had been linear (Drazin and Johnson). The nonlinear interaction of waves is characterized by phase shifts. The solitons occur as $t \to \pm\infty$ and interact in this special way.

At a large time, the difference in wave speed creates two separate waves. The solution then consists of two waves. This suggests one solitary with wave speed 16 and the other with wave speed 4. Therefore, introduce $(x-16t)$ and $(x-4t)$ to find the solution's asymptotic behavior. As time goes to infinity, the solution suggests a wave moving with a velocity of 16 units, an amplitude of 8 units, and a phase shift of $\ln(3/2)$. Then, as time goes to negative infinity, a phase shift of $-\ln(3/2)$ will be similar to the wave moving with a velocity of 4 units and an amplitude of 2 units. As the time goes to infinity, the faster wave will move forward by $\ln(3/2)$, and the shorter wave will move backward by $\ln(3/2)$ (Munteanu, L., & Donescu, S, pages 101-103).

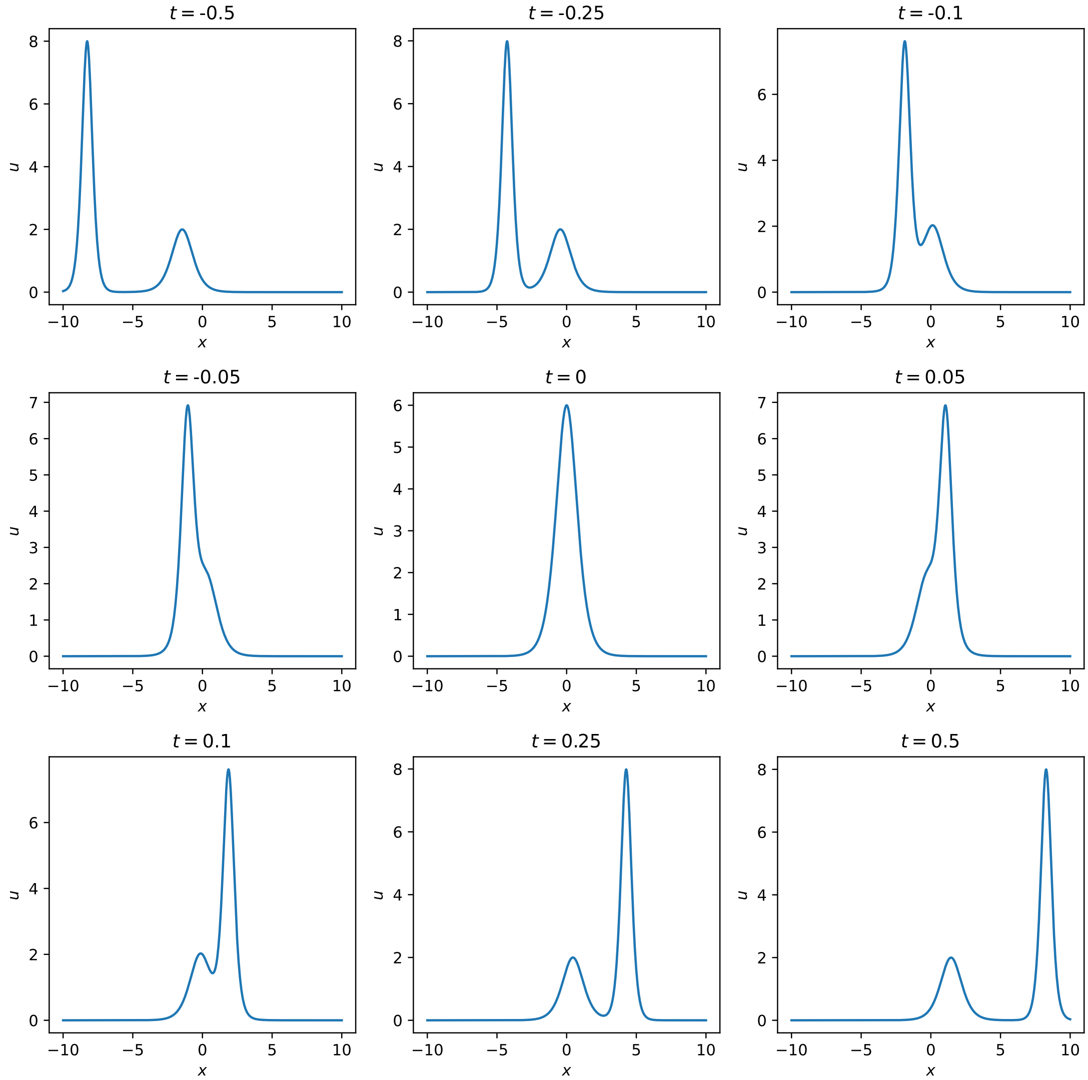

Figure 2: A plot of the propagation of two solitons with time in two-soliton solution system.

3.3 Non Linear Schrödinger Equation

The solution of the nonlinear Schrödinger equation is a hyperbolic secant function. First, consider the amplitude squared to extract the probability density (due to the wave nature of the function):

$$|\Psi|^2 = \Phi_0^2 \operatorname{sech}^2\left(\sqrt{\frac{x - u_e t}{L_e}}\right)$$

This results in a secant squared function similar to the solutions for the KdV equation (see Figures 1 and 2). The number of solitons depends on the parameters P and Q. For example, in optical physics, the number of solitons N is given by

$$N^2 = \frac{x_0^2 k_0 n}{2\eta / k_0 n n_2 |A_m|^2} = \frac{x_0^2 Q}{2P}$$

where P and Q are given by:

$$P = \frac{1}{2k_0 n} \qquad Q = \frac{k_0 n n_2 |A_m|^2}{2\eta_0}$$

Here, k_0 is the wave number, n and n_2 are the initial and final index of refraction of the medium, A_m is the maximum amplitude, and η_0 is the impedance of free space.

Evidently, the number of solitons is related to the properties of both the medium and the wave used.

4 Properties and Behaviour

4.1 Stability

The stability of solitons comes from a delicate balance between nonlinearity and dispersion in the medium through which they travel. (Steven Abe, 2006).

> The theory of linear dispersive equations predicts that waves should spread out and disperse over time. (Terence Tao, 2008)

The nonlinear term in the KdV equation is uu_x, which makes an important contribution to the stability of solitons. When the nonlinear term is removed from the formula, it leaves $u_t + u_{xxx} = 0$. The result is dispersion. Waves disperse and spread out as time increases. If the dispersive term is removed it leaves $u_t + 6uu_x = 0$, the result is breaking, and the waves become concentrated.

4.2 Interaction and Propagation

Consider the KdV equation's solution from IST when $N = 1$, which is $u(x,t) = \operatorname{sech}(x - ct)$. The parameter is the speed of wave c. When it is positive, the wave propagates to the left; converselty, when negative it propagates to the right. The soliton propagates in a single direction as time varies.

As shown in Figure 2, two waves travel in the same direction, and when $t = 0$, there is only one wave in the figure. Now consider that a solution is formed by the superposition of two solitary solutions (this is not a soliton solution, as mentioned earlier), where one wave has an amplitude of 8 and the other 2.

That is, $u * (x,t) = 2\operatorname{sech}^2(x - 4t) + 8\operatorname{sech}^2(x - 16t)$. When two waves interact at $t = 0$, a single wave with an amplitude of 10 is formed, unlike the wave with an amplitude of 6 in the two-soliton solution.

4.2.1 Non-Destructive Interactions

Unlike regular waves that might merge or dissipate upon colliding, solitons pass through each other without permanent change to their shape or velocity. This property is a hallmark of solitons and is a direct consequence of their nonlinear nature.

4.2.2 Phase Shift

Although solitons emerge from interactions unchanged in terms of amplitude and velocity, they can experience a phase shift. This means that their positions after interaction are shifted relative to where they would have been if they had travelled without interacting. The phase shift is a crucial feature in the study of solitons, as it reflects the integral effect of the nonlinear interaction.

5 Applications

Soliton theory has been used to study many significant practical problems in fluid dynamics, plasma, nonlinear optics, astrophysics, and molecular biology. In fibre optics, for example, the concept of solitons has been an ongoing topic of research for long-distance digital signal transmission.

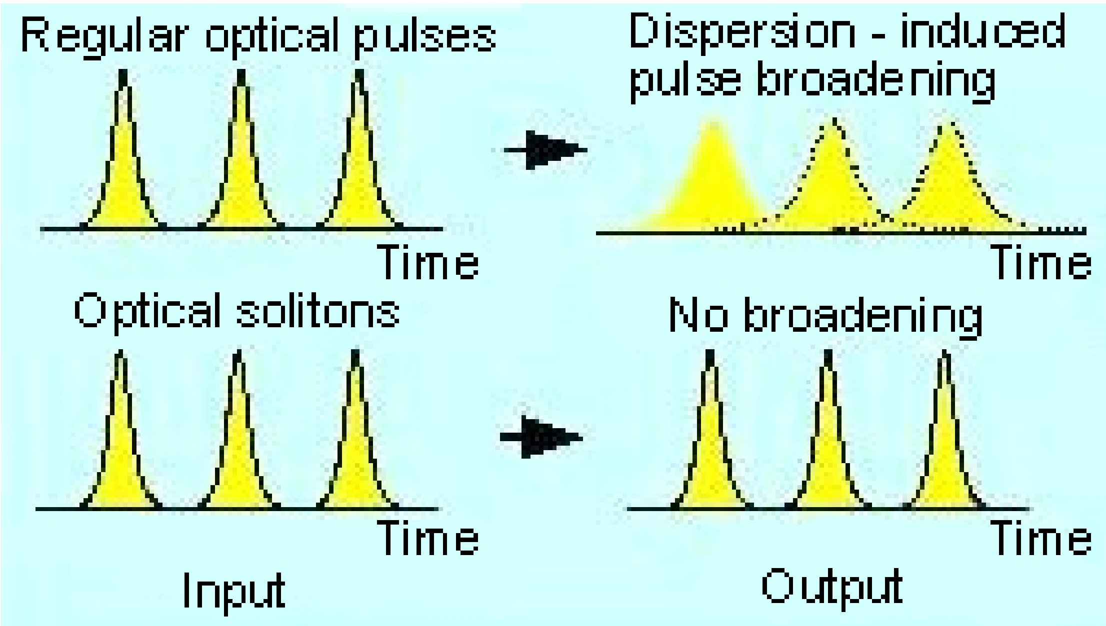

Figure 3: Stability of regular pulses compared to soliton pulses. Soliton pulses are much more stable and do not disperse over long distances (Research Institute of Electrical Communications, Tohoku University, n.d.)

As described in the above sections, solitons are stable due to the nonlinearity term in the NLS equation. On the contrary, regular pulses will spread out and overlap each other, resulting in possible information loss as shown in Figure 4. The equation that models these pulses in optical fibers is the nonlinear Schrödinger equation (3). In this specific case, the constants P and Q are proportional to the refractive index of the medium and other important constants. As shown in the derivation of the solutions for the NLS equation, specific conditions have to be met in order to produce these solitons, which can be difficult to achieve in the real world.

In hydrodynamics, solitons can accurately model tsunamis and tidal bores due to the fact that they can travel long distances without changing shape (Marin, 2009). In quantum and particle physics, solitons can be applied to explain the behavior and properties of elementary particles. Specifically, in the context of quantum field theory, solitons are non-perturbative solutions to the field equations. These solutions represent stable and localized packets of energy that can't be dispersed. In essence, they can be viewed as "knots" of energy that exhibit particle-like properties which can interact with other particles and fields.

This concept can be applied to the theory of quarks. The idea is that solution solutions to the equations governing quantum chromodynamics, the theory that describes the strong force that holds quarks together, could provide a possible explanation for the confinement of these quarks and the fact that they are never isolated. Lastly, solitons also appear in matter physics, particularly in bose-einstein condensates (BEC). For very cold temperatures (close to absolute zero), the wavefunction of the BEC system can be modeled by the NLS equation which therefore enables solitonic solutions. In experimental results, these solitons can be distinguished as "bright" and "dark" solitons (Becker et al, 2008). Unlike the regular "bright" solitons shown in Figure 3, "dark" solitons are stable dips in intensity as shown below.

These experimental results not only confirm the prediction of solitons in the NLS equation but also open paths for further research between the nonlinearity and dispersion of these quantum systems.

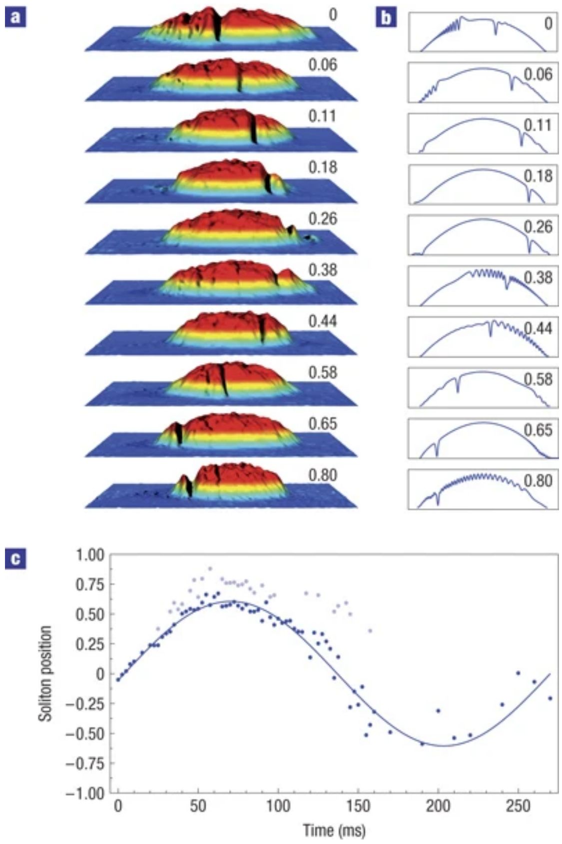

Figure 4: Propogation of dark solitons in the BEC. (Taken from Figure 2 in Becker et al, 2008)

6 Conclusion

We have derived solutions for the KdV equation and the nonlinear Schrödinger's equation and showed that solitonic solutions exist for specific initial conditions. The mathematical framework, particularly the concept of inverse scattering transform, provides insights into the solutions of these equations. Additionally, solitons arise from the balance between the dispersion term and the non-linear term that holds the wave together. We also plotted the one-soliton and two-soliton solutions that arise from the KdV equation and showed that because of the nonlinearity, the collision of the two solitons does not result in superposition. On the contrary, they seem to collide and return back to their original shape, thus illustrating their stability.

Beyond applications in hydrodynamics, quantum, and matter physics, solitons have paved the way for breakthroughs in optical fiber communications. Due to their stability, they can travel long distances without dispersing and losing valuable information, thus making them a intriguing topic of research. Beyond practical uses, solitons continue to be valuable in abstract and mathematical areas such as quantum chromodynamics, where they are theorized to play an important role in the understanding of the strong force.

7 References

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Appendix

Listing 1: Inverse Scattering Transform Methodology

The inverse scattering transform begins with a transformation of u called a Miura transformation (initially found by R. Miura, 1968) (Gardner, C.S., Greene, J.M., Kruskal, M.D. and Miura, R.M. (1967) Method for Solving Korteweg-Devries Equation. Physical Review Letters, 19, 1095-1097.).

The transformation is: $u = \lambda - v^2 - v_x$ for a constant λ. If v satisfies $v_t + 6(-v^2)v_x + v_{xxx} = 0$, then the u given by the transformation is a solution to the KdV equation. This method works by reversing this implication, that is to say: if there is a known solution u to the KdV equation, then it can be used to solve for v given the equation of the transformation. Letting $v = x$, the problem becomes one of solving:

$$\Psi_{xx} + u\Psi = \lambda\Psi$$

This equation is also known as the time-independent Schrödinger equation for the Sturm-Liouville problem. Taking a brief historical detour, at the time of the research being conducted, interest was peaked in such equations. This is the quantum mechanical equation for a particle in a potential of -u. A lot of research had gone into the Schrödinger equation, and much was known about their solutions. Thus, it is a pleasant surprise that this equation can allow us to find soliton solutions.

Overview and Remarks: The inverse scattering transform has 3 main steps. Each are quite lengthy so a summary of the important results of each step is provided along with a description of the methodology. The ultimate goal is to construct time-dependent solutions. For further reading of the underlying mathematical machinery see (Abel, S. 2006 Pages 77-106)

Step 1 - Disassembly: Begin with initial data $u(x, 0)$ playing the role of the potential. This yields a different eigenfunction Ψ for each λ. Each eigenfunction describes the scattering of a particle off the potential. This description is given through the asymptotic values of Ψ at $x \to \pm\infty$, which is contained in transmission and reflection coefficients. These coefficients make up what is known as initial scattering data. To illustrate, recall that a fourier transform of $u(x, t)$ is given by:

$$\hat{u}(k, t) = \int_{-\infty}^{\infty} u(x, t)e^{-ikx}dx$$

In the scattering data, λ is analogous to k in the Fourier transform, and the transform itself has been replaced with determining the components of v from u.

Step 2 - Time Evolution: A very nice feature of the KdV equation simplifies this step greatly is the following: for any u which satisfies the KdV equation,

λ is time-independent and thus constant in the realm of time evolutions of the scattering data. Therefore, all that needs to be done is to evolve the scattering data.

Step 3 - Inverse Scattering: As the name suggests, the last step requires the reassembly of $u(x,t)$ at t. It is already known that this is possible for the Sturm Liouville equation.

Listing 2: Non-Linear Schrödinger Equation Solution Derivation

As described in section 2.3.1, we look for solutions of the form

$$\Psi = \Phi(x,t)e^{i\theta(x,t)} \tag{8}$$

Where Φ and θ are real functions.

Substituting (8) into (3) and seperating the real and imaginary parts leads to the following system of equations:

$$-\Phi\theta_t + P\Phi_{xx} - P\Phi\theta_x^2 + Q\Phi^3 = 0 \tag{9}$$

$$\Phi_t + 2P\Phi_x\theta_x + P\Phi\theta_{xx} = 0 \tag{10}$$

We look for solutions in which both the carrier wave θ and the envelope Φ are permanent profile solutions with different propagation velocities u_p for θ and u_e for Φ.

$$\Phi(x,t) = \Phi(x - u_e t) \tag{11}$$

$$\theta(x,t) = \theta(x - u_p t) \tag{12}$$

Substituting these 2 relations into (9) and (10) gives:

$$u_p\Phi\theta_x + P\Phi_{xx} - P\Phi\theta_x^2 + Q\Phi^3 = 0 \tag{13}$$

$$-u_e\Phi_x + 2P\Phi_x\theta_x + P\Phi\theta_{xx} = 0 \tag{14}$$

Multiplying (14) by Φ and integrating gives:

$$-\frac{u_e\Phi^2}{2} + P\Phi^2\theta_x = C$$

For spatially localized solutions, we assume that Φ and Φ_x tend to zero as $|x| \to \infty$. Therefore $C = 0$.

$$P\Phi^2\theta_x = \frac{u_e\Phi^2}{2} \tag{15}$$

Isolating for θ_x gives:

$$\theta_x = \frac{u_e}{2P} \tag{16}$$

Integrating gives:

$$\theta = \frac{u_e}{2P}(x - u_p t) \tag{17}$$

Substituting (17) into (13) and multiplying by $P\Phi_x$ gives:

$$\frac{u_e u_p}{2P}P\Phi\Phi_x + P^2\Phi_x\Phi_{xx} - \frac{u_e^2}{4P}P\Phi\Phi_x + PQ\Phi^3\Phi_x = 0$$

Integrating the above equation gives:

$$\frac{P^2}{2}\Phi_x^2 + V_{\text{eff}}(\Phi) = 0 \tag{18}$$

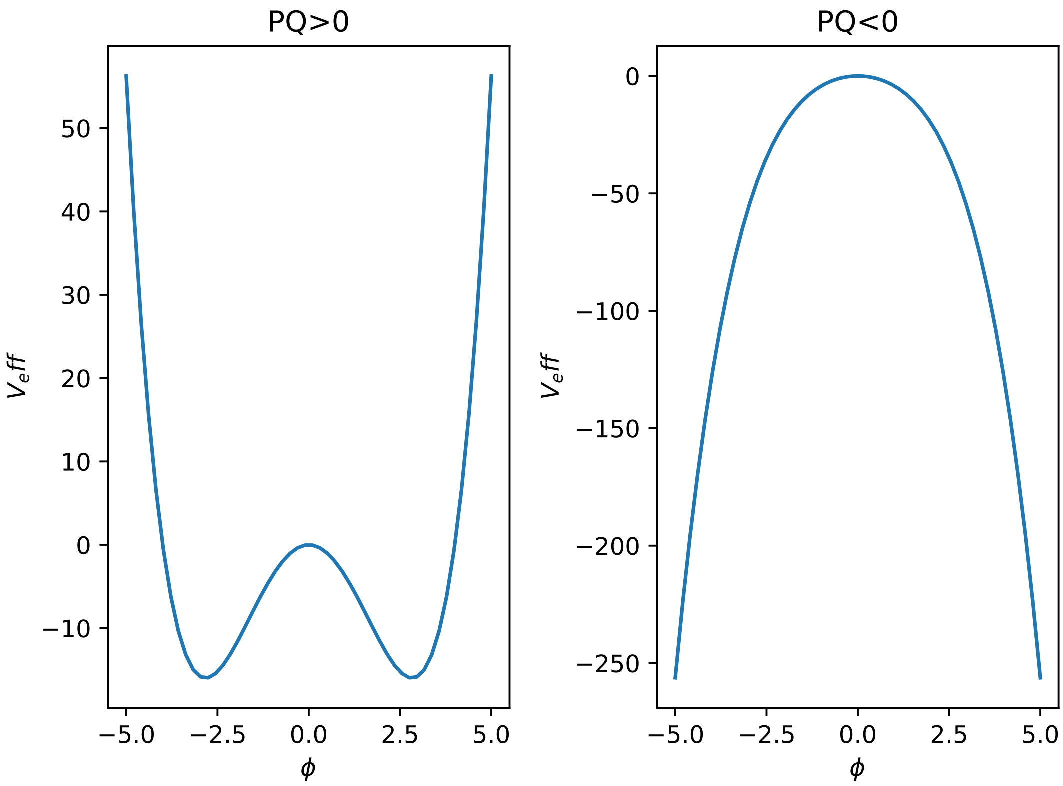

Figure 5: Pseudopotential Plot

where

$$V_{\text{eff}}(\Phi) = \frac{PQ}{4}\Phi^4 - \frac{(u_e^2 - 2u_eu_p)}{8}\Phi^2$$

As Φ is a real number, $\Phi_x^2 \geq 0$, so (18) implies that $V_{\text{eff}} \leq 0$ for all Φ values which correspond to a solution.

Figure 5 shows the plot of the pseudo-potential for positive and negative values of PQ. For positive values of PQ, the plot shows a bell-shaped curve with Φ bounded between two intervals. The motion stars at $\Phi = 0$, reaches $\Phi = \Phi_0$ where

$$\Phi_0 = \sqrt{\frac{(u_e^2 - 2u_eu_p)}{2PQ}}$$

and then comes back to the starting point. On the other hand, for negative values of PQ, there are no such bounds. Therefore, this potential shows that the NLS equation can only have localized soliton-like solutions if $PQ > 0$.

Integrating (18) using the change of variable $\Phi = \Phi_0 \operatorname{sech}(v)$ gives:

$$\Psi = \Phi_0 \operatorname{sech}\left[\sqrt{\frac{Q}{2P}}\Phi_0(x - u_et) + \operatorname{arcsech}\left(\frac{\Phi(0,0)}{\Phi_0}\right)\right] \tag{19}$$

Listing 3: Python Code Used to Generate Figure 1

```
import numpy as np
import matplotlib.pyplot as plt
t1 = -1.0
t2 = 0
t3 = 1.0
x = np.linspace(-10,10,1000)
y1 = 2* 1/(np.cosh(x-4*t1))**2
y2 = 2*1/(np.cosh(x-4*t2))**2
y3 = 2*1/(np.cosh(x-4*t3))**2

fig, ax = plt.subplots()
plt.plot(x,y1,label = '$t = -1.0$')
plt.plot(x,y2,label = '$t = 0$')
plt.plot(x,y3,label = '$t = 1.0$')

plt.xlabel("$x$")
plt.ylabel("$u(x,t)$")

leg = ax.legend();

plt.savefig('Single_Soliton_Sol.eps', format='eps')
```

Listing 4: Python Code Used to Generate Figure 2

```
import numpy as np
import matplotlib.pyplot as plt

t = [-0.5,-0.25,-0.1,-0.05,0,0.05,0.1,0.25,0.5]
x = np.linspace(-10,10,1000)
y= [0,0,0,0,0,0,0,0,0]
for i in range(9):
    y[i] = 12*(3+4*np.cosh(2*x-8*t[i])+np.cosh(4*x-64*t[i]))/
    (3*np.cosh(x-28*t[i]) + np.cosh(3*x-36*t[i]) )**2


fig, ax = plt.subplots(3,3, figsize=(10,10))
count=0
for i in range(3):
    for j in range(3):
        ax[i,j].plot(x,y[count])
        ax[i,j].set_title('$t=$' + str(t[count]))
        ax[i,j].set_xlabel("$x$")
        ax[i,j].set_ylabel("$u$")
        count+= 1
fig.tight_layout(pad=1)
plt.show()
fig.savefig('Two_Soliton_Prop.eps', format='eps')
```

Listing 5: Python Code Used to Generate Figure 5

```
import matplotlib.pyplot as plt
import numpy as np
v = np.linspace(-5,5,50)
y1 = 1/4*v**4 - 4*v**2
y2 = -1/4*v**4 - 4*v**2
plt.subplot(1,2,1)
plt.plot(v,y1)
plt.xlabel("$\phi$")
plt.ylabel("$V_eff$")
plt.title("PQ>0")
plt.subplot(1,2,2)
plt.xlabel("$\phi$")
plt.ylabel("$V_eff$")
plt.plot(v, y2)
plt.title("PQ<0")
plt.tight_layout(pad=1)

plt.savefig('Potential.eps', format='eps')
```